Available online at www.sciencedirect.com

SCIENCE DIRECT®

Ann. I. H. Poincaré – PR 41 (2005) 1101–1123

ANNALES DE L'INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES

www.elsevier.com/locate/anihpb

Green kernel estimates and the full Martin boundary for random walks on lamplighter groups and Diestel–Leader graphs ☆

Sara Brofferio [1], Wolfgang Woess *

Institut für Mathematik C, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria

Received 5 April 2004; accepted 22 December 2004

Abstract

We determine the precise asymptotic behaviour (in space) of the Green kernel of simple random walk with drift on the Diestel–Leader graph $\mathsf{DL}(q,r)$, where $q,r \geqslant 2$. The latter is the horocyclic product of two homogeneous trees with respective degrees $q+1$ and $r+1$. When $q=r$, it is the Cayley graph of the wreath product (lamplighter group) $\mathbb{Z}_q \wr \mathbb{Z}$ with respect to a natural set of generators. We describe the full Martin compactification of these random walks on DL-graphs and, in particular, lamplighter groups. This completes previous results of Woess, who has determined all minimal positive harmonic functions.

Résumé

On détermine le comportement asymptotique précis (dans l'espace) du noyau de Green de la marche aléatoire simple avec dérive sur le graphe de Diestel–Leader $\mathsf{DL}(q,r)$, où $q,r \geqslant 2$. Ce graphe est le produit horocyclique de deux arbres homogènes de degrés $q+1$ et $r+1$, respectivement. Quand $q=r$, il s'agit du graphe de Cayley du produit en couronne (« lamplighter group ») $\mathbb{Z}_q \wr \mathbb{Z}$ par rapport à un ensemble naturel de générateurs. On décrit la compactification de Martin complète de ces marches aléatoires sur les graphes DL, et en particulier, les groupes du « lamplighter ». Ceci complète les résultats précédents de Woess, qui a déterminé les fonctions harmoniques minimales.

MSC: 60J50; 05C25; 20E22; 31C05; 60G50

Keywords: Lamplighter group; Wreath product; Diestel–Leader graph; Random walk; Martin boundary; Harmonic functions

☆ Supported by European Commission, Marie Curie Fellowship HPMF-CT-2002-02137 and partially by FWF (Austrian Science Fund) project P15577.

* Corresponding author.
E-mail address: woess@weyl.math.tu-graz.ac.at (W. Woess).

[1] Current address: Laboratoire de Mathématiques, Université Paris-Sud, bâtiment 425, 91405 Orsay cedex, France.

0246-0203/$ – see front matter
doi:10.1016/j.anihpb.2004.12.004

1. Introduction

Consider the additive group $\mathbb{Z}$ of all integers as a two-way-infinite road where at each point there is a lamp that may be switched on in one of q different intensities (states) $\in \mathbb{Z}_q = \{0, \ldots, q-1\}$, the group of integers modulo q. At the beginning, all lamps are in state 0 (switched off), and a lamplighter starts at some point of $\mathbb{Z}$. He chooses at random among the following actions (or a suitable combination thereof): he can move to a neighbour point in $\mathbb{Z}$, or he can change the intensity of the lamp at the actual site to a different state. As the process evolves, we have to keep track of the position $k \in \mathbb{Z}$ of the lamplighter and of the finitely supported configuration $\eta : \mathbb{Z} \to \mathbb{Z}_q$ that describes the states of all lamps. The set $\mathbb{Z}_q \wr \mathbb{Z}$ of all pairs (η, k) of this type carries the structure of a semi-direct product of $\mathbb{Z}$ with the additive group $\mathcal{C}$ of all configurations, on which $\mathbb{Z}$ acts by translations. This is often called the *lamplighter group*; the underlying algebraic construction is the *wreath product* of two groups.

Random walks on lamplighter groups have been a well-studied subject in recent years, see Kaimanovich and Vershik [18] and Kaimanovich [17] (Poisson boundary $\equiv$ bounded harmonic functions), Lyons, Pemantle and Peres [20], Erschler [12], Revelle [24], Bertacchi [3] (rate of escape), Grigorchuk and Żuk [13], Dicks and Schick [7], Bartholdi and Woess [2] (spectral theory), Saloff-Coste and Pittet [22,23], Revelle [25] (asymptotic behaviour of transition probabilities), and Woess [28] (positive harmonic functions).

Here, we shall deal with Green kernel asymptotics and positive harmonic functions. Let us briefly outline in general how this is linked with *Martin boundary theory* of Markov chains. Consider an arbitrary infinite (connected, locally finite) graph X (e.g., a Cayley graph of a finitely generated group) and the stochastic transition matrix $P = (p(x,y))_{x,y \in X}$ of a random walk Z_n on X. That is, Z_n is an X-valued random variable, the position of the random walker at time n, subject to the Markovian transition rule

$$\mathsf{Pr}[Z_{n+1} = y \mid Z_n = x] = p(x,y).$$

The n-step transition probability

$$p^{(n)}(x,y) = \mathsf{Pr}[Z_n = y \mid Z_0 = x], \quad x, y \in X,$$

is the (x,y)-entry of the matrix power P^n, with $P^0 = I$, the identity matrix. The *Green kernel* is

$$G(x,y) = \sum_{n=0}^{\infty} p^{(n)}(x,y), \quad x, y \in X.$$

This is the expected number of visits in the point y, when the random walk starts at x. We always consider random walks that are *irreducible* and *transient*, which amounts to

$$0 < G(x,y) < \infty \quad \text{for all } x, y \in X.$$

Renewal theory in a wide sense consists in describing the asymptotic behaviour in space of $G(x,y)$, when x is fixed and y tends to infinity (or dually, y is fixed and x tends to infinity). If we fix a reference point $o \in X$, then the *Martin kernel* is

$$K(x,y) = G(x,y)/G(o,y), \quad x, y \in X.$$

If we have precise asymptotic estimates in space of the Green kernel, then we can also determine the *Martin compactification*. This is the smallest metrizable compactification of X containing X as a discrete, dense subset, and to which all functions $K(x,\cdot)$, $x \in X$, extend continuously. The *Martin boundary* $\mathcal{M} = \mathcal{M}(P)$ is the ideal boundary added to X in this compactification. Thus, $\mathcal{M}$ consists of the "directions of convergence" of $K(x,y)$, when $y \to \infty$. Its significance is that it leads to a complete understanding of the cone $\mathcal{H}^+ = \mathcal{H}^+(P)$ of *positive harmonic functions*. A function $h : X \to \mathbb{R}$ is called *harmonic*, or P-*harmonic*, if

$$h = Ph, \quad \text{where } Ph(x) = \sum_y p(x,y)h(y).$$