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Poisson point processes attached to symmetric diffusions

Masatoshi Fukushima [a,*], Hiroshi Tanaka [b]

[a] *Department of Mathematics, Faculty of Engineering, Kansai University, Suita 564-8680, Japan*
[b] *1-4-17-104 Miyamaedaira, Miyamae-ku, Kawasaki 216-0006, Japan*

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Abstract

Let a be a non-isolated point of a topological space S and $X^0=(X_t^0, 0\leqslant t<\zeta^0, P_x^0)$ be a symmetric diffusion on $S_0=S\setminus\{a\}$ such that $P_x^0(\zeta^0<\infty, X^0_{\zeta^0-}=a)>0$, $x\in S_0$. By making use of Poisson point processes taking values in the spaces of excursions around a whose characteristic measures are uniquely determined by X^0, we construct a symmetric diffusion $\widetilde{X}$ on S with no killing inside S which extends X^0 on S_0. We also prove that such a process $\widetilde{X}$ is unique in law and its resolvent and Dirichlet form admit explicit expressions in terms of X^0.

Résumé

Etant donné un point a non isolé d'un espace topologique S, nous considérons une diffusion symétrique $X^0=(X_t^0, P_x^0)$ dans $S_0=S\setminus\{a\}$ telle que $P_x^0(\zeta^0<\infty, X^0_{\zeta^0-}=a)>0$ et $P_x^0(\zeta^0<\infty, X^0_{\zeta^0-}\in S_0)=0$ pour tout $x\in S_0$ où ζ^0 est la durée de vie. En utilisant les processus de Poisson ponctuels des excursions partant de a dont les mesures caractéristiques sont déterminées par X^0, nous construirons une diffusion symétrique $\widetilde{X}$ dans S qui est une extension de X^0 et dont les trajectoires ne disparaisssent pas à l'intérieur de S. Nous montrons aussi qu'une telle extension est unique en loi et que sa résolvente et sa forme de Dirichlet admettent les expressions explicites en terme de X^0.

Keywords: Symmetric diffusion; Poisson point process; Excursions; Entrance law; Energy functional; Dirichlet form

* Corresponding author.
E-mail address: fuku@ipcku.kansai-u.ac.jp (M. Fukushima).

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1. Introduction

Let S be a locally compact separable metric space and a be a non-isolated point of S. We put $S_0 = S \setminus \{a\}$. The one point compactification of S is denoted by S_Δ. When S is compact already, Δ is added as an isolated point. Let m be a positive Radon measure on S_0 with $\mathrm{Supp}[m] = S_0$. m is extended to S by setting $m(\{a\}) = 0$.

We assume that we are given an m-symmetric diffusion $X^0 = (X_t^0, P_x^0)$ on S_0 with life time ζ^0 satisfying the following four conditions:

A.1 $P_x^0\big(\zeta^0 < \infty, X^0_{\zeta^0-} \in \{a\} \cup \{\Delta\}\big) = P_x^0(\zeta^0 < \infty), \quad \forall x \in S_0.$

We define the functions $\varphi(x), u_\alpha(x), \alpha > 0$, of $x \in S_0$ by

$$\varphi(x) = P_x^0(\zeta^0 < \infty, X^0_{\zeta^0-} = a), \quad u_\alpha(x) = E_x^0(\mathrm{e}^{-\alpha\zeta^0}; X^0_{\zeta^0-} = a).$$

A.2 $\varphi(x) > 0,\ \forall x \in S_0$.
A.3 $u_\alpha \in L^1(S_0; m),\ \forall \alpha > 0$.
A.4 $u_\alpha \in C_b(S_0),\ G_\alpha^0(C_b(S_0)) \subset C_b(S_0),\ \alpha > 0,$

where G_α^0 is the resolvent of X^0 and $C_b(S_0)$ is the space of all bounded continuous functions on S_0.

By making use of excursion-valued Poisson point processes whose characteristic measures are uniquely determined by X^0, or to be a little more precise, by piecing together those excursions which start from a and return to a and then possibly by adding the last one that never returns to a, we shall construct in §4 of the present paper a process $\widetilde{X}$ on S satisfying

(1) $\widetilde{X}$ is an m-symmetric diffusion process on S with no killing inside S,
(2) $\widetilde{X}$ is an extension of X^0: the process on S_0 obtained from $\widetilde{X}$ by killing upon the hitting time of a is identical in law with X^0.

We call a process $\widetilde{X}$ on S satisfying (1), (2) a *symmetric extension of* X^0.

We shall also prove in §5 that, under conditions A.1, A.2 for the given m-symmetric diffusion X^0 on S_0, its symmetric extension is unique in law, satisfies condition A.3 automatically and admits the resolvent expressible as

$$G_\alpha f(x) = G_\alpha^0 f(x) + u_\alpha(x) \cdot G_\alpha f(a), \quad x \in S_0, \quad G_\alpha f(a) = \frac{(u_\alpha, f)}{\alpha(u_\alpha, \varphi) + L(m_0, \psi)},$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(S_0; m)$ and $L(m_0, \psi)$ is the energy functional in Meyer's sense [21] of the X^0-excessive measure $m_0 = \varphi \cdot m$ and X^0-excessive function $\psi = 1 - \varphi$.

Furthermore the associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(S; m)$ will be seen in §5 to have the following simple expression; if we denote by $\mathcal{F}_e$ its extended Dirichlet space, then

$$\mathcal{F}_e = \{w = u_0 + c\varphi:\ u_0 \in \mathcal{F}_{0,e},\ c \text{ constant}\}, \qquad \mathcal{F} = \mathcal{F}_e \cap L^2(S; m),$$
$$\mathcal{E}(w, w) = \mathcal{E}(u_0, u_0) + c^2\mathcal{E}(\varphi, \varphi), \qquad \mathcal{E}(\varphi, \varphi) = L(m_0, \psi),$$

where $(\mathcal{F}_{0,e}, \mathcal{E})$ is the extended Dirichlet space for the given diffusion X^0.

In §6, we shall present four examples. Example 6.1 concerns the uniqueness of the symmetric extension of the one-dimensional absorbing Brownian motion.

Example 6.2 treats the case where S_0 is a bounded open subset of $\mathbb{R}^d$ $(d \geqslant 1)$, $S = S_0 \cup \{a\}$ is the one point compactification of S_0 and X^0 is the absorbing Brownian motion on S_0. In this case, $\varphi(x) = 1,\ x \in S_0$. The resulting Dirichlet form on $L^2(S; m)$ (m is the Lebesgue measure on S_0 extended to S by $m(\{a\}) = 0$) is given by