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Large deviations for invariant measures of stochastic reaction–diffusion systems with multiplicative noise and non-Lipschitz reaction term

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Abstract

In this paper we prove a large deviations principle for the invariant measures of a class of reaction–diffusion systems in bounded domains of $\mathbb{R}^d$, $d \geqslant 1$, perturbed by a noise of multiplicative type. We consider reaction terms which are not Lipschitz-continuous and diffusion coefficients in front of the noise which are not bounded and may be degenerate. This covers for example the case of Ginzburg–Landau systems with unbounded and possibly degenerate multiplicative noise.

Résumé

Dans cet article on prouve un principe de grandes déviations pour les mesures invariantes de systèmes de réaction–diffusion stochastiques dans des domaines bornés de $\mathbb{R}^d$, $d \geqslant 1$, perturbés par un bruit multiplicatif. On considère des termes de réaction qui ne sont pas lipschitziens et des coefficients de diffusion qui ne sont pas bornés et peuvent être dégénérés. Ceci s'applique par exemple au cas de systèmes de Ginzburg–Landau avec bruit multiplicatif non borné et éventuellement dégénéré.

MSC: 60F10; 60H15

Keywords: Large deviations principle; Stochastic partial differential equations; Invariant measures; Multiplicative noise

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1. Introduction

In this paper we are dealing with the long-term behavior of the stochastic reaction–diffusion system

$$\begin{cases} \frac{\partial u_i}{\partial t}(t,\xi) = \mathcal{A}_i u_i(t,\xi) + f_i(\xi, u_1(t,\xi),\dots,u_r(t,\xi)) \\ \quad + \varepsilon \sum_{j=1}^r g_{ij}(\xi, u_1(t,\xi),\dots,u_r(t,\xi)) Q_j \frac{\partial w_j}{\partial t}(t,\xi), \quad t \geqslant 0, \quad \xi \in \overline{\mathcal{O}}, \\ u_i(0,\xi) = x_i(\xi), \quad \xi \in \overline{\mathcal{O}}, \qquad \mathcal{B}^i u_i(t,\xi) = 0, \quad t \geqslant 0,\ \xi \in \partial\mathcal{O},\ 1 \leqslant i \leqslant r, \end{cases} \tag{1.1}$$

with $\varepsilon > 0$. Here $\mathcal{O}$ is a bounded open set of $\mathbb{R}^d$, with $d \geqslant 1$, having a C^∞ boundary. For each $i = 1,\dots,r$

$$\mathcal{A}_i(\xi, D) = \sum_{h,k=1}^d \frac{\partial}{\partial \xi_h}\left(a^i_{hk}(\xi)\frac{\partial}{\partial \xi_k}\right) - \alpha_i, \quad \xi \in \overline{\mathcal{O}}. \tag{1.2}$$

The constants α_i are strictly positive, the coefficients a^i_{hk} are taken in $C^\infty(\overline{\mathcal{O}})$ and the matrices $a^i(\xi) := [a^i_{hk}(\xi)]_{hk}$ are non-negative and symmetric, for each $\xi \in \overline{\mathcal{O}}$, and fulfill a uniform ellipticity condition, that is

$$\inf_{\xi \in \overline{\mathcal{O}}} \langle a^i(\xi)h, h\rangle \geqslant \lambda_i |h|^2, \quad h \in \mathbb{R}^d,$$

for some positive constants λ_i. Finally, the operators $\mathcal{B}^i$ act on $\partial\mathcal{O}$ and are assumed either of Dirichlet or of co-normal type.

The mapping $f := (f_1,\dots,f_r) : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}^r$ is only locally Lipschitz-continuous and has polynomial growth. The mapping $g := [g_{ij}] : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is Lipschitz-continuous, without any assumption of boundedness and non-degeneracy.

The linear operators Q_j are bounded on $L^2(\mathcal{O})$ and may be taken to be equal to the identity operator in case of space dimension $d = 1$. The noisy perturbations $\partial w_j/\partial t$ are independent cylindrical Wiener processes on a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

For example, in the case of space dimension $d = 1$ and $r = 2$, we can deal with systems of the following type

$$\begin{cases} \frac{\partial u_1}{\partial t} = \frac{\partial}{\partial \xi}\left(a_1 \frac{\partial u_1}{\partial \xi}\right) - \alpha_1 u_1 - c_1 u_1^{2k+1} + f_1(u_1,u_2) + \left\langle g_1(u_1,u_2), \frac{\partial w}{\partial t}\right\rangle, \\ \frac{\partial u_2}{\partial t} = \frac{\partial}{\partial \xi}\left(a_2 \frac{\partial u_2}{\partial \xi}\right) - \alpha_2 u_2 - c_2 u_2^{2k+1} + f_2(u_1,u_2) + \left\langle g_2(u_1,u_2), \frac{\partial w}{\partial t}\right\rangle, \\ u_i(0,\xi) = x_i(\xi), \quad \xi \in \mathcal{O}, \qquad u_i(t,\xi) = \eta_i u_i(t,\xi) + (1-\eta_i)\frac{\partial u_i}{\partial \xi}(t,\xi), \quad \xi \in \partial\mathcal{O}, \end{cases}$$

where a_i are positive functions in $C^1(\overline{\mathcal{O}})$, $\eta_i \in \{0,1\}$, α_i and c_i are positive constants (in fact α_i can be taken zero in the case of Dirichlet boundary conditions, that is if $\eta_i = 1$), $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 function having linear growth, with $f(0) = 0$ and $Df(0)$ diagonal, and $g = (g_1, g_2) : \mathbb{R}^2 \to \mathcal{L}(\mathbb{R}^2)$ is any Lipschitz continuous function such that $g(0)$ either vanishes or is diagonal invertible and such that

$$\|g(\sigma)\|_{\mathcal{L}(\mathbb{R}^2)} \leqslant c\left(1 + |\sigma|^\gamma\right), \quad \sigma \in \mathbb{R}^2,$$

with

$$2k + 1 > (1 + 6\gamma) \vee 2.$$

In particular, if g is bounded in the reaction term we can take any power $2k + 1 \geqslant 3$.

In [2] it is proved that for any $\varepsilon > 0$ and $p \geqslant 1$ system (1.1) admits a unique global solution $u^x_\varepsilon \in L^p(\Omega; C([0,T]; E))$, where E is the space of continuous functions on $\overline{\mathcal{O}}$ with values in $\mathbb{R}^r$, and for each initial datum $x \in E$ and $a > 0$ the family of probability measures $\{\mathcal{L}(u^x_\varepsilon(t)\}_{t \geqslant a}$ is tight in $(E, \mathcal{B}(E))$. In particular,