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Games and Economic Behavior 50 (2005) 89–106

GAMES and Economic Behavior

www.elsevier.com/locate/geb

A family of ordinal solutions to bargaining problems with many players ☆

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Received 3 November 2003

Available online 8 December 2004

Abstract

A solution to bargaining problems is ordinal when it is covariant with respect to order-preserving transformations of utility. Shapley has constructed an ordinal, symmetric, efficient solution to three-player problems. Here, we extend Shapley's solution in two directions. First, we extend it to more than three players. Second, we show that this extension lends itself to the construction of a continuum of ordinal, symmetric, efficient solutions. The construction makes use of ordinal path-valued solutions that were suggested and studied by O'Neil et al. [Games Econ. Behav. 48 (2004) 139–153].

Keywords: Bargaining problems; Ordinal utility; Bargaining solutions

1. Introduction

1.1. Ordinal solutions

A bargaining problem is described here, as in Nash's (1950) bargaining theory, by the set of all utility vectors that arise from possible agreements. A solution is a function that selects for each problem a vector of utilities.

☆ A PowerPoint presentation of this article is available at http://www.tau.ac.il/~samet.

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0899-8256/$ – see front matter
doi:10.1016/j.geb.2004.08.001

The utility functions in Nash's theory are assumed to be derived from the von Neumann–Morgenstern representation of preferences. This representation is determined up to linear positive transformations of the utility functions. Therefore, any two problems obtained from each other by such transformations should be considered equivalent. Thus, a solution in this theory must be covariant with respect to such transformation. Namely, it should assign to any two equivalent problems the same solution, up to the required transformation. Indeed, one of the axioms which characterizes Nash's solution spells explicitly this requirement.

Suppose, that contrary to Nash's theory, no assumption is made on the utility functions other than that they represent preferences (i.e. the more preferred outcome has a higher utility). In this case the presentation of preferences is determined up to order-preserving (i.e. monotonically increasing) transformations of utility functions. Hence, a solution in this bargaining theory should be covariant with respect to these transformations. We say that such a solution is *ordinal*.

1.2. Shapley's solution for three players

Shapley (1969) has shown that there is no ordinal, symmetric, and efficient solution for bargaining problems of two players. However, he has constructed such a solution for three-player problems (see Shubik, 1982).[1]

The construction is based on the following observation. Suppose that $a = (a_1, a_2, a_3)$ is the disagreement point of a bargaining problem with a Pareto surface S. Then there exists a unique point $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, such that the points,

$$(a_1, \bar{x}_2, \bar{x}_3), \qquad (\bar{x}_1, a_2, \bar{x}_3), \quad \text{and} \quad (\bar{x}_1, \bar{x}_2, a_3),$$

are all in S. In the terminology of Kalai and Smorodinsky (1975) the point a is the ideal point for $\bar{x}$.[2] Reversing the order we say that the point $\bar{x}$ is the *ground* point for a. (See Fig. 1.)

The relation between a point and its unique ground point is ordinal. Thus, assigning to each problem the ground point of its disagreement point is an ordinal solution. This solution is also symmetric, but it is not on the Pareto surface of the problem.

To fix this latter deficiency Shapley used this solution iteratively, applying it in each step to the problem with the same Pareto surface S, and a disagreement point which is the solution obtained in the previous step. The sequence of points generated this way can be shown to converge to a point on the Pareto surface, which is the desired solution.

The construction of Shapley's solution hinges on both the existence and the uniqueness of the ground point $\bar{x}$ for any given a. For more than three players the construction cannot be carried out since the uniqueness of a ground point is not guaranteed, as was shown by Sprumont (2000). However, Safra and Samet (2004) proved for any number of players the existence of at least one ground point for each point a. They used the *set* of ground

[1] Recently, Kıbrıs (2003) has proposed an axiomatization of the three-player Shapley solution.

[2] Kalai and Smorodinsky (1975) defined the ideal point for a feasible disagreement point. However, the feasibility assumption is not used in their definition, and therefore it can be applied also to infeasible points like $\bar{x}$ in this example.