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Some asymptotic results for sums of dependent random variables, with actuarial applications

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Abstract

This paper establishes some asymptotic results for sums of dependent random variables, in the presence of heavy-tailedness conditions. We demonstrate how the derived results can be used to approximate functionals of sums of dependent random variables for which the analytic expression is too cumbersome to work with and which are of major importance in actuarial applications. Numerical illustrations are provided to assess the quality of the asymptotic approximations.

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1. Introduction

Many quantities of relevance in actuarial science involve sums of dependent random variables. For example, one may think of the value-at-risk of a stochastically discounted life annuity, or the stop-loss premium for the aggregate claim amount of a number of interrelated policies. Therefore, distribution functions of sums of dependent random variables are of particular interest. Typically such distribution functions are of a complex form. Consequently, in order to compute functionals of sums of dependent random variables, approximation methods are often indispensable.

In case the dependence structure between the elements of the random sum is known, one could use Monte Carlo simulation to obtain empirical distribution functions. However, this is typically a time-consuming approach, in particular if we want to approximate tail probabilities, which would require an excessive number of simulations. Therefore, alternative methods need to be explored.

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By now a rich literature is available on the use of conditional expectations and the concept of *comonotonicity* to obtain bounds in convex order for sums of dependent random variables; the interested reader is referred to Rogers and Shi (1995), Kaas et al. (2000) and Dhaene et al. (2002a,b). While these bounds in convex order have proven to be good approximations in case the distribution of the random sum is light-tailed or moderately heavy-tailed, they perform worse when the heavy-tailedness (the volatility, for the (log)normal case) increases; see e.g., Section 4.4 of Dhaene et al. (2002b).

In actuarial applications, it is often merely the tail of the distribution function that is of interest. Indeed, one may think of (tail-)value-at-risk or expected shortfall estimations. Therefore, approximations for functionals of sums of dependent random variables may alternatively be obtained through the use of asymptotic relations. Though asymptotic results are valid *at* infinity, they may as well serve as approximations *near* infinity.

This paper establishes some asymptotic results for the tail probability of sums of dependent random variables, in the presence of heavy-tailedness conditions. In particular, we establish an asymptotic result for the randomly weighted sum of a sequence of non-negative numbers. Furthermore, we establish, under two different sets of conditions, an asymptotic result for the randomly weighted sum of a sequence of independent random variables that consist of a deterministic and a random component. Throughout, the random weights are products of i.i.d. random variables and thus exhibit an explicit dependence structure. Next, we present three actuarial applications that demonstrate how the derived asymptotic results can be employed to approximate certain functionals of sums of dependent random variables. To explore the quality of the asymptotic approximations, we provide several numerical illustrations that compare the asymptotic approximation values to Monte Carlo simulated values. For one of the illustrations, we have also included the approximation results obtained by two other approximation methods based on comonotonicity and moment matching.

The outline of the paper is as follows: in Section 2, we introduce some notational conventions and provide some preliminaries for heavy-tailed distributions. In Section 3, we present the asymptotic results. Section 4 provides a first application of the obtained asymptotic results, concerning the evaluation of stop-loss premiums and quantiles for general stochastically discounted loss reserves. In Section 5, we present a second application that focuses specifically on IBNR loss reserves. Section 6 presents a third application, which considers the problem of setting an initial provision in such a way that the probabilities of ruin in year i, $i = 1, \dots, n$, are sufficiently small. Numerical results are presented in Section 7. Proofs of the theorems and the comonotonic and moment matching approximation formulas have been gathered in Appendices A and B.

2. Preliminaries for heavy-tailed distributions

First we introduce some notational conventions. For a random variable (r.v.) X with a distribution function (d.f.) F, we denote its tail probability by $\overline{F}(x) = 1 - F(x) = \mathbb{P}(X > x)$. For two independent r.v.'s X and Y with d.f.'s F and G supported on $(-\infty, +\infty)$, we write by $F * G(x) = \int_{-\infty}^{+\infty} F(x-t)\,\mathrm{d}G(t)$, $-\infty < x < +\infty$, the convolution of F and G, and by $F^{*n} = F * \cdots * F$ the n-fold convolution of F. By $F \otimes G$ we denote the d.f. of XY.

Throughout, unless otherwise stated, all limit relations are for $x \to +\infty$. Let $a(x) \geq 0$ and $b(x) > 0$ be two infinitesimals, satisfying

$$l_1 \leq \liminf_{x\to+\infty} \frac{a(x)}{b(x)} \leq \limsup_{x\to+\infty} \frac{a(x)}{b(x)} \leq l_2.$$

We write $a(x) = O(b(x))$ if $l_2 < +\infty$, $a(x) = o(b(x))$ if $l_2 = 0$ and $a(x) \asymp b(x)$ if both $l_2 < +\infty$ and $l_1 > 0$. We write $a(x) \lesssim b(x)$ if $l_2 = 1$, $a(x) \gtrsim b(x)$ if $l_1 = 1$ and $a(x) \sim b(x)$ if both $l_2 = 1$ and $l_1 = 1$. We say that $a(x)$ and $b(x)$ are weakly equivalent if $a(x) \asymp b(x)$, and say that $a(x)$ and $b(x)$ are (strongly) equivalent if $a(x) \sim b(x)$.

A r.v. X or its d.f. F is said to be *heavy-tailed to the right* if $\mathbb{E}[e^{\gamma X}] = +\infty$ for any $\gamma > 0$. Below we introduce some important classes of heavy-tailed distributions. A d.f. F supported on $[0, +\infty)$ belongs to the *subexponential*