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Graphical potential games

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Abstract

We study the class of potential games that are also graphical games with respect to a given graph G of connections between the players. We show that, up to strategic equivalence, this class of games can be identified with the set of Markov random fields on G. From this characterization, and from the Hammersley–Clifford theorem, it follows that the potentials of such games can be decomposed into local potentials.

Keywords: Potential games; Graphical games

1. Introduction

Potential games form an important class of strategic interactions. They includes fundamental interactions such as Cournot oligopolies (see, e.g., Monderer and Shapley, 1996), congestion games (see, e.g., Monderer and Shapley, 1996 or Rosenthal, 1973), routing games (see, e.g., Rosenthal, 1973) and many others. The above mentioned interactions are frequently *local* in nature. Namely, there exists an underlying graph such that the payoff of a player depends on her own strategy and on the strategies of her neighbors, but does not depend on the strategies of the opponents who are not neighbors. For instance, the locality of the interaction could be geographical: In routing games, the outcome of a driver depends only on her own route and the routes that were chosen by drivers who are geographically close to her. In a Cournot oligopoly

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where transportation costs are high, a firm is competing only with firms which are geographically close (for instance, this is the case with natural gas market; see Victor et al., 2006). The idea of the locality of an interaction is captured by the notion of a *graphical game*, introduced by Kearns et al. (2001). These games and similar ones are also sometimes called *network games*; see Jackson and Zenou (2012) for an extensive survey.

The goal of this paper is to understand the class of graphical potential games. First, we address the following questions: What characterizes the potential function of a graphical potential game? What characterizes the payoffs of the players in a potential graphical game? In Theorems 4.1, 4.4 and 4.5 we provide a complete answer to these questions. We show that

(1) The potential function of a graphical potential game can be expressed as an additive function of *local potentials*, where each local potential corresponds to a maximal clique in the underlying graph, and the value of the local potential is determined by the strategies of the players in the maximal clique only. This condition is necessary for a potential game to be graphical, and every such potential is the potential of some graphical game.
(2) Up to strategically equivalent transformations (see Definition 4.3), the payoff of a player is the sum of the local potentials of the cliques to which she belongs.

The proof of these results is achieved by showing that, for a fixed graph G, the set of potentials of graphical games on G can be identified with the set of *Markov random fields* on G (see Section 3.4). The latter are a well studied class of probability distributions with certain graphical Markov properties. Having established this correspondence, the Hammersley–Clifford theorem (Hammersley and Clifford, 1968), a classical result on Markov random fields, yields the above characterization of graphical potential games.

An extended online version of this paper additionally includes applications to better-response dynamics and to infinite games (Babichenko and Tamuz, 2014).

1.1. Related literature

Potential games and graphical games are both fundamental classes of games (see, e.g., Nisan, 2007). There has recently been a growing interest in the intersection of these two classes, because many interesting potential games have a graphical structure (see e.g., Bilò et al., 2011 or Bimpikis et al., 2014), and many interesting graphical games have a potential function (see e.g., Auletta et al., 2011, Bramoullé et al., 2014). However, to the best of our knowledge, this paper is the first to fully characterize the intersection of these two classes. Similar questions are studied independently by Ortiz (2015).

The connection between graphical potential games and Markov random fields was previously observed for specific cases; see Auletta et al. (2011), who establish a connection between logit dynamics in a particular class of graphical potential coordination games and Glauber dynamics in the Markov random field of the Ising model. We show that this connection is much more general and extends to all graphical potential games.

Another intriguing connection between graphical games and Markov random fields was established by Kakade et al. (2003), who show that every correlated equilibrium of every graphical game is a Markov random field. Daskalakis and Papadimitriou (2006) use Markov random fields to compute pure Nash equilibria in graphical games.