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Fast convergence in evolutionary models: A Lyapunov approach ☆

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Abstract

Evolutionary models in which N players are repeatedly matched to play a game have "fast convergence" to a set A if the models both reach A quickly and leave A slowly, where "quickly" and "slowly" refer to whether the expected hitting and exit times remain bounded when N tends to infinity. We provide simple and general Lyapunov criteria which are sufficient for reaching quickly and leaving slowly. We use these criteria to determine aspects of learning models that promote fast convergence.

JEL classification: C73; C69; D83

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1. Introduction

The study of stochastic stability in evolutionary models focuses on the long-run outcomes of various sorts of adjustment processes that combine best response or learning dynamics with

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mutations, errors, or other sorts of random fluctuations. Because the stochastic terms make these systems ergodic, they have a unique invariant distribution which corresponds to their long-run outcome, and since these outcomes typically single out a single equilibrium they provide a way to do equilibrium selection, as in Kandori et al. (1993) and Young (1993). However, the long-run outcome is only relevant if it is reached in a reasonable amount of time in populations of the relevant size, and this is not the case if agents intend to play a best response to the current state and the stochastic term arises from a constant probability of error, as here the expected time for the population to shift to the risk-dominant equilibrium is exponentially large in the population (Ellison, 1993).

For this reason, there has been considerable interest in understanding when the long run outcome is reached "quickly" in the sense that the expected number of time periods to reach a neighborhood of the selected outcome is bounded above, independent of the population size. Past work in economics has used one of two methods for showing that this occurs: either an argument using coupled Markov processes as in Ellison (1993), or a two-step approach of first showing that the associated deterministic, continuous time, mean field has a global attractor, and then showing that the discrete-time stochastic system behaves approximately the same way when N is large, as in Kreindler and Young (2013, 2014). We say more about these papers below.[1]

This paper provides a simple Lyapunov condition for quickness that covers this past work. We also provide a complementary Lyapunov condition under which the process leaves the target set "slowly," in the sense that the probability of getting more than ϵ away from the set in any fixed time goes to zero as the population size increases. As this latter property seems necessary for convergence to the target set to be interesting, we only say that there is "fast convergence" when both conditions hold. By providing a unified approach to proving fast convergence, we highlight the connection between them; this may provide intuition about other settings where the same result will apply. Our conditions are also relatively tractable and portable, which lets us prove that there is fast convergence in a number of new cases that are more complicated than those already in the literature. As one example of this, Section 2 presents a model of local interaction with a small-world element, where players interact both with their neighbors and with randomly drawn members of the whole population; here neither of the past techniques can be applied.

Section 3 presents the general model, which is based on a collection of time-homogeneous Markov chains $S^N = \{S^N(t) : t = 0, \frac{1}{N}, \frac{2}{N}, \ldots\}$ with finite state spaces Ω_N, where N indexes the number of players in the population. These Markov chains may track for example the play of each player at each location of a network. We then define functions ϕ_N on Ω_N that map to a space $\mathcal{X}$ of fixed dimension, and consider the processes given by $X^N(t) = \phi_N(S^N(t))$. For example, these processes can describe the share of agents using each of a finite number of pure strategies in a game.

Section 4 presents a pair of general results that use a Lyapunov function V to provide sufficient conditions for "reaching quickly" and "leaving slowly." Proposition 1 says roughly that the system reaches a subset A of $\mathcal{X}$ quickly if the expectation of $V(X^N(t))$ decreases by at least a fixed rate $c > 0$ whenever the state is outside of A, that is,

$$E\left[V\left(X^N\left(t+\frac{1}{N}\right)\right) - V\left(X^N(t)\right) \middle| S^N(t)\right] \le -\frac{c}{N} \text{ when } X^N(t) \notin A.$$

[1] In mathematical biology hitting times for evolutionary processes have been characterized by diffusion approximations as in Ethier and Kurtz (1986) and Ewens (2004) and have been explicitly calculated for birth-death processes as in Nakajima and Masuda (2015).