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Journal of Economic Theory 148 (2013) 2666–2695

JOURNAL OF Economic Theory

www.elsevier.com/locate/jet

# Higher order game dynamics ☆

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Received 28 June 2012; final version received 30 July 2013; accepted 31 July 2013

Available online 12 August 2013

**Abstract**

Continuous-time game dynamics are typically first order systems where payoffs determine the growth rate of the players' strategy shares. In this paper, we investigate what happens beyond first order by viewing payoffs as higher order forces of change, specifying e.g. the acceleration of the players' evolution instead of its velocity (a viewpoint which emerges naturally when it comes to aggregating empirical data of past instances of play). To that end, we derive a wide class of higher order game dynamics, generalizing first order imitative dynamics, and, in particular, the replicator dynamics. We show that strictly dominated strategies become extinct in $n$-th order payoff-monotonic dynamics $n$ orders as fast as in the corresponding first order dynamics; furthermore, in stark contrast to first order, weakly dominated strategies also become extinct for $n \geqslant 2$. All in all, higher order payoff-monotonic dynamics lead to the elimination of weakly dominated strategies, followed by the iterated deletion of strictly dominated strategies, thus providing a dynamic justification of the well-known epistemic rationalizability process of Dekel and Fudenberg [7]. Finally, we also establish a higher order analogue of the folk theorem of evolutionary game theory, and we show that convergence to strict equilibria in $n$-th order dynamics is $n$ orders as fast as in first order.


*JEL classification:* C61; C62; C72; C73

☆ Part of this work was carried out when Panayotis Mertikopoulos was with the Economics Department of the École Polytechnique, Paris, France, and we would like to thank the École for its hospitality and generous support. The authors also gratefully acknowledge financial support from the French National Agency for Research (grant ANR-10-BLAN 0112), the GIS "Sciences de la Décision", and the Pôle de Recherche MSTIC under grant no. C-UJF-LACODS-2012.

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0022-0531/$ – see front matter 
http://dx.doi.org/10.1016/j.jet.2013.08.002

*Keywords:* Game dynamics; Higher order dynamical systems; (Weakly) dominated strategies; Learning; Replicator dynamics; Stability of equilibria

## 1. Introduction

Owing to the considerable complexity of computing Nash equilibria and other rationalizable outcomes in non-cooperative games, a fundamental question that arises is whether these outcomes may be regarded as the result of a dynamic learning process where the participants "accumulate empirical information on the relative advantages of the various pure strategies at their disposal" [24, p. 21]. To that end, numerous classes of game dynamics have been proposed (from both a learning and an evolutionary "mass-action" perspective), each with its own distinct set of traits and characteristics – see e.g. the comprehensive survey by Sandholm [30] for a most recent account.

Be that as it may, there are few rationality properties that are shared by a decisive majority of game dynamics. For instance, if we focus on the continuous-time, deterministic regime, a simple comparison between the well-known replicator dynamics [35] and the Smith dynamics [33] reveals that game dynamics can be imitative (replicator) or innovative (Smith), rest points might properly contain the game's Nash set or coincide with it [16], and strictly dominated strategies might become extinct [29] or instead survive [17]. In fact, negative results seem to be much more ubiquitous: there is no class of uncoupled game dynamics that always converges to equilibrium [13] and weakly dominated strategies may survive in the long run, even in simple $2 \times 2$ games [28,37].

From a mathematical standpoint, the single unifying feature of the vast majority of game dynamics is that they are first order dynamical systems. Interestingly however, this restriction to first order is not present in the closely related field of optimization (corresponding to games against nature): as it happens, the second order "heavy ball with friction" method studied by Alvarez [1] and Attouch et al. [2] has some remarkable optimization properties that first order schemes do not possess. In particular, by interpreting the gradient of the function to be maximized as a physical, Newtonian force (and not as a first order vector field to be tracked by the system's trajectories), one can give the system enough energy to escape the basins of attraction of local maxima and converge instead to the *global* maximum of the objective function (something which is not possible in ordinary first order dynamics).

This, therefore, begs the question: *can second (or higher) order dynamics be introduced and justified in a game theoretic setting?* And if yes, *do they allow us to obtain better convergence results and/or escape any of the first order impossibility results?*

The first challenge to overcome here is that second order methods in optimization apply to *unconstrained* problems, whereas game dynamics must respect the (constrained) structure of the game's strategy space. To circumvent this constraint, Flåm and Morgan [8] proposed a heavy-ball method as in Attouch et al. [2] above, and they enforced consistency by projecting the orbits' velocity to a subspace of admissible directions when the updating would lead to inadmissible strategy profiles (say, assigning negative probability to an action). Unfortunately, as is often the case with projection-based schemes (see e.g. Sandholm et al. [31]), the resulting dynamics are not continuous, so even basic existence and uniqueness results are hard to obtain.

On the other hand, if players try to improve their performance by aggregating information on the relative payoff differences of their pure strategies, then this cumulative empirical data is not constrained (as mixed strategies are). Thus, a promising way to obtain a well-behaved second