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Spatial dynamics and convergence: The spatial AK model ☆

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Abstract

We study the optimal dynamics of an AK economy where population is uniformly distributed along the unit circle. Locations only differ in initial capital endowments. Spatio-temporal capital dynamics are described by a parabolic partial differential equation. The application of the maximum principle leads to necessary but non-sufficient first-order conditions. Thanks to the linearity of the production technology and the special spatial setting considered, the value function of the problem is found explicitly, and the (unique) optimal control is identified in feedback form. Despite constant returns to capital, we prove that the spatio-temporal dynamics, induced by the willingness of the planner to give the same (detrended) consumption over space and time, lead to convergence in the level of capital across locations in the long-run.

JEL classification: C60; O11; R11; R12; R13

Keywords: Economic growth; Spatial dynamics; Optimal control; Partial differential equations

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1. Introduction

Optimal and market allocation of economic activity across space has always been a central issue in economic theory from the seminal work of Hotelling [11]. Recently, some authors have studied the optimal spatial allocation of economic activity in dynamic settings with capital accumulation. To our knowledge, Brito [4] is the first attempt to fully characterize the corresponding optimal spatio-temporal dynamics, followed by Brock and Xepapadeas [5] and Boucekkine et al. [3]. This research is surveyed by Desmet and Rossi-Hansberg [7].

Factor mobility turns out to be crucial: Brito and Boucekkine et al. consider frictionless capital mobility while Brock and Xepapadeas invoke a spatial externality without capital mobility. In the former, the production function exhibits decreasing returns: capital flows from regions with low marginal productivity of capital to regions with high marginal productivity. As a result, capital spatio-temporal dynamics are shown to be driven by a partial differential equation (PDE) of the form:

$$\frac{\partial k}{\partial t}(t,z) - \frac{\partial^2 k}{\partial z^2}(t,z) = F\big(k(t,z),z\big) - c(t,z), \tag{1}$$

where z is the spatial position (which could be a position in the real line as in Boucekkine et al. [3], or in the unit circle as in this article). Per capita consumption and capital, $c(t,z)$ and $k(t,z)$ respectively vary in time and space. The production function $F(\cdot,\cdot)$ can depend explicitly on space. The specific nature of the equation comes from the term $\frac{\partial^2 k}{\partial z^2}(t,z)$ which captures capital flows across space as explained in Section 2: this makes the problem infinite-dimensioned.

Both Brito and Boucekkine et al. have attempted to solve spatial Ramsey models where capital follows Eq. (1). Both have used a straight line as a model of space and have used an adapted maximum principle to derive the corresponding first-order conditions, in particular the adjoint equation which is a PDE too. As explained in Boucekkine et al., the maximum principle yields an ill-posed system of PDE equations. The problem arises from the specific generated adjoint equation which, coupled with the associated transversality condition, does not allow to prove neither the existence nor uniqueness of an optimal control (ill-posedness). In particular, potential multiplicity of solutions is the key problem faced by both Brito and Boucekkine et al., who have ended up restricting the model and/or the set of optimal solutions to get rid of this problem: Boucekkine et al. restrict utility functions to be linear and Brito identifies a special type of solutions (called *travelling waves*).

In this paper, we consider the *AK* production function case and we model space as a circle. Even if the linear production function simplifies the adjoint equation and the space is bounded, the problem mentioned above still remains as we will show. Precisely, we show that ill-posedness is due to the fact that the first-order optimality conditions found by Boucekkine et al., specified for our AK problem, are necessary but not sufficient to determine the optimal solution; as a result, other "irrelevant" solutions to these conditions do emerge. This makes a big difference with respect to the standard finite-dimensioned AK model (without space) where the first-order conditions are also sufficient. The key tool to reach these results is the use of a dynamic programming method well adapted to the infinite dimensionality of the problem. After rewriting the problem in a suitable infinite dimensional space, we exploit the linearity of the production function and the spatial setting (that's the circle as a compact manifold without boundary) to identify an explicit value function, which in turn allows us to solve the problem in feedback form and then to find explicitly the optimal control. The methodology is described in the last paragraph of Section 2, Appendix A gives the related details. We prove that the unique solution to the dynamic program-