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Convergence in models with bounded expected relative hazard rates ☆

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Abstract

We provide a general framework to study stochastic sequences related to individual learning in economics, learning automata in computer sciences, social learning in marketing, and other applications. More precisely, we study the asymptotic properties of a class of stochastic sequences that take values in [0, 1] and satisfy a property called "bounded expected relative hazard rates." Sequences that satisfy this property and feature "small step-size" or "shrinking step-size" converge to 1 with high probability or almost surely, respectively. These convergence results yield conditions for the learning models in [13,35,7] to choose expected payoff maximizing actions with probability one in the long run.

JEL classification: D81; D83

Keywords: Hazard rate; Individual learning; Social learning; Two-armed bandit algorithm; Dynamic system; Stochastic approximation; Submartingale; Convergence

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1. Introduction

Stochastic sequences arising in the analysis of several models in economics often exhibit expected hazard rates that are proportional to the sequence's current value. For instance, models of technology adoption often satisfy that the change in the fraction of a population that adopts a new technology is proportional to the product of the current fraction of adopters and the current fraction of non-adopters (see, e.g., [42]). This follows from the assumption that diffusion of technology requires non-adopters to observe adopters in order to learn about the new technology. A similar reasoning applies to models in other disciplines, such as Bass' celebrated model of new product growth (see, e.g., [2,16]) and selection models in biological evolution (see, e.g., [29]). As we discuss below, models of individual and social learning provide another class of examples for stochastic sequences with expected hazard rates that are proportional to the sequences' current value. In these models, the sequences represent the probability of choosing optimal actions.

The analysis of such models usually concerns the question whether a new technology or a product gets fully adopted, a certain type takes over in a biological selection process, or an optimal action is played almost surely in the long run. Towards this end, this paper provides general conditions on expected hazard rates of a bounded stochastic sequence that guarantee the convergence to the upper bound. Here, the sequence is interpreted as a fraction of a certain type or the probability of playing an optimal action at any point in time. This paper thus provides conditions that guarantee that, in the long run, a certain type takes over the whole group of types or only optimal actions are chosen, as illustrated in the applications discussed below.

It turns out that constraints on the *relative hazard rates* of a stochastic sequence, i.e., the proportions of the hazard rates to the values of the sequence,[1] provide helpful conditions for the convergence to the upper bound. In contrast to the deterministic case, in a stochastic framework, lower bounds for the relative hazard rates are not sufficient for almost sure convergence. For example, in the case of technology adoption, full adoption might fail as the new technology may be completely abandoned at some point in time by chance, or adoption rates may drop too fast. The analysis below reveals that if the underlying submartingale moves in small or shrinking steps, convergence to the upper bound holds, nevertheless. Thus, in the long run, new technologies are used or optimal actions chosen if adoption or learning occurs in small or shrinking steps.

The first main result of this paper, Theorem 2.1, analyzes the asymptotical properties of a sequence that changes with *small step-size* and satisfies weak bounds on its relative hazard rates. Theorem 2.1 asserts that the probability of *convergence to optimality*, i.e., the event that the stochastic sequence converges to the upper bound, is arbitrarily high for sequences with sufficiently small step-size. This result allows us to obtain novel convergence results in different contexts, including, for instance, the models of individual and social learning that we discuss below. A limitation of Theorem 2.1 is that the question of how small the step-size needs to be in order to achieve any given probability of convergence to 1 is usually directly related to the probability measure of the underlying probability space. In applications, however, this probability measure is assumed to be unknown. This issue is addressed by Theorem 2.2 and Corollary 2.1, which provide sufficient conditions for achieving convergence to optimality almost surely under an extra condition that may be interpreted as requiring an arbitrary *shrinking step-size* over time.

These results can be applied to the analysis of several models of boundedly rational learning (see, e.g., [13,35,7]). In models of individual learning, in every period individuals choose

[1] Formally, if the values of the sequence are denoted by $\{P_t\}_{t\in\mathbb{N}_0}$, then the corresponding relative hazard rates are defined as $(P_{t+1} - P_t)/((1 - P_t)P_t)$ for all $t \in \mathbb{N}_0$.