Contracts versus salaries in matching: A general result

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Abstract

It is shown that a matching market with contracts can be embedded into a matching market with salaries under weaker conditions than substitutability of contracts. In particular, the result applies to the recently studied problem of cadet-to-branch matching. As an application of the embedding result, a new class of mechanisms for matching markets with contracts is defined that generalize the firm-proposing deferred acceptance algorithm to the case where contracts are unilateral substitutes for firms.
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The matching with contracts framework (Hatfield and Milgrom, 2005; Kelso and Crawford, 1982; Roth, 1984; Fleiner, 2003) is a key model in recent market design research. It has been successfully applied to the matching of cadets to branches in the United States Military Academy (Sönmez and Switzer, 2013) and the Reserve Officer Training Corps (Sönmez, 2013) as well as to the design of affirmative action matching mechanisms in school choice (Kominers and Sönmez, forthcoming). Despite this practical success, some theoretical questions about the model have not been satisfyingly answered. In particular, it is not clear to what extent the model is more general than the job matching model of Kelso and Crawford (1982).

In the discussion of their job matching model with salaries, Kelso and Crawford (1982) already discuss the possibility to extend their model to allow for arbitrary "endogenous job char-

acteristics", i.e. multidimensional contracts. Moreover, they give an informal discussion of how to reduce this seemingly more general model to their original framework. As long as agents' preferences allow for "a well-behaved utility possibility frontier" and contracts are "negotiated efficiently" a parameterization of the utility possibility frontiers for matched firms and workers can be interpreted as salaries and the original results extend to the more general framework. Recently, it was shown that this informal argument can be made precise and that the assumption that contracts are substitutes for firms is important for the argument (Echenique, 2012). Under this assumption, there exists an embedding that assigns to each market with contracts a corresponding market with salaries such that the set of stable allocations of the market is invariant under the embedding. Moreover, the gross substitutability condition that is the key assumption of the analysis of Kelso and Crawford (1982) is satisfied in the market with salaries. This result can be extended to many-to-many models of matching with contracts (Kominers, 2012).

The embedding results show that, under the assumption of substitutability, the matching with contracts model is essentially not more general than the matching model with salaries. Nevertheless, substitutability is not the most general condition for which the key results of the theory of many-to-one matching with contracts hold. The main results of the theory can be proved under the weaker condition of unilateral substitutability, respectively under the even weaker condition of bilateral substitutability (Hatfield and Kojima, 2010).[1] Furthermore, these weaker substitutability conditions play a central role in recent market design applications (Sönmez and Switzer, 2013; Sönmez, 2013; Kominers and Sönmez, forthcoming).

We extend the result of Echenique (2012) and show that a market where contracts are unilateral substitutes for firms can be embedded into a market with salaries where workers are gross substitutes for firms. In particular, the result applies to the cadet-to-branch matching problem studied by Sönmez and Switzer (2013). We show that under a weaker notion of embeddability, an embedding is possible even when contracts are bilateral substitutes for firms. For this purpose, we introduce a natural condition for the demand in market with salaries, *net substitutability*, that guarantees the convergence of a descending auction to a stable allocation. We then show that a market where contracts are bilateral substitutes for firms can be embedded into a market with salaries where workers are net substitutes for firms. Both results are the most general that we can hope for. We show that, for the embedding method proposed in this paper, unilateral substitutability is necessary for an embedding into a market with gross substitutability and that bilateral substitutability is necessary for an embedding into a market with net substitutability. In this sense, the results of this paper clarify to what extent the model with contracts is more general than the model with salaries.

The main technical contribution of this paper is a new embedding technique that does not rely on separability[2] of firm preferences, i.e. the property that a firm's ranking of contracts with a given worker is independent of the contracts it signs with other workers. Separability in many-to-one markets with contracts is implied by substitutability but not by unilateral substitutability (Hatfield and Kojima, 2010). Thus, under unilateral substitutability we cannot use the embedding construction of Echenique (2012) and Kominers (2012). The underlying observation behind our embedding is that firm preferences can be modified such that they become essen-

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[1] Unilateral substitutability is sufficient for the existence of a worker-optimal stable allocation. Unilateral substitutability and the law of aggregate demand are sufficient for the rural hospitals theorem, group-strategy-proofness and weak Pareto optimality of the cumulative offer mechanism for the workers. Bilateral substitutability guarantees the existence of stable allocations.

[2] Other authors starting with Roth (1984) use the term "Pareto separability".