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Racing under uncertainty: Boundary value problem approach

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Abstract

This paper explores the continuous time and continuous space model of racing under uncertainty put forward by Budd, Harris, and Vickers [4] and allows for potentially asymmetric players. To prove the existence of Markov perfect equilibria, I use a boundary value problem formulation which is novel to the dynamic competition literature. In addition, by providing a new and intuitive definition of the *pivot* of an equilibrium, I show that equilibrium strategies exhibit the *discouragement effect* similar to that of Harris and Vickers [10] but under a more general class of the cost functions.

JEL classification: C62; C72; C73

Keywords: Dynamic competition; Continuous time game; Discouragement effect; Boundary value problem

1. Introduction

A race is a contest between two or more competitors who exert effort to win a prize. Sport contests, such as bicycle races, golf tournaments and basketball championships, are the most popular forms of races. Races studied in economic theory include patent races and contests for job promotion.

Despite its importance, the theoretical literature on dynamic competition has been relatively sparse. Harris and Vickers [10] is the pioneering paper with a model in discrete state-space, i.e., the distance between players can take discrete values. In their model, they prove that at least one equilibrium exists and characterize its properties. The current paper uses a continuous time,

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continuous state-space model based on Harris and Vickers [10]'s tug-of-war model to address the question of the existence and explore the properties of equilibrium strategies. The existence and characterization theorems apply for general cases with asymmetric players and discounting.

In the current model, two players compete for a final reward. The reward is won by the first player who achieves a given distance over his rival. At any moment in the race, each player exerts effort which influences the distance between him and his rival. The distance follows a Brownian motion with a drift that depends on the efforts of the players. The flow cost of effort functions are strictly convex. I consider the set of Markovian Perfect Equilibria (MPE) in which equilibrium strategies of the players are conditioned only on the current distance between them.

I apply the theory of boundary value problems for systems of second-order differential equations developed in Hartman [12] to show the existence of MPE. Moreover, in all MPE of the current model, the strategies share basic properties with the equilibrium strategies of the discrete state space model by Harris and Vickers [10]. The MPE strategies exhibit a *discouragement effect*: the players exert high effort only when they are close to each other. When a player is positioned far behind by his rival, he reduces his effort given his slim chance of winning. The rival who is further ahead therefore faces less competition and can safely reduce his effort. He will do so if he does not discount the future. A larger distance between the two players thus discourages both players. However, Harris and Vickers [10] show the discouragement effect only for the case in which two players have the same cost function and both of them do not discount the future. My paper generalizes this result to cases with asymmetric players and with discounting. Specifically, I adapt the definition of the *pivot* of an MPE from Harris and Vickers [10] to these cases with asymmetric players: the pivot is the single-crossing point of the effort functions of the two players. This new definition of the pivot allows me to show that the discouragement effect is still obtained in these cases.

Under quadratic cost functions and no discounting, this continuous time and continuous state-space framework delivers a closed-form solution of the MPE strategies.[1] This close-form solution illustrates how the discouragement effect works and how the degree of uncertainty, final reward, and cost of effort functions, affect the intensity of the effect.

Budd, Harris, and Vickers [4], the first paper to formulate Harris and Vickers [10] in continuous time and continuous space, solve a similar model using boundary value representations. However, their method only applies when the discount rate r goes to infinity. Moscarini and Smith [15] are the first to analytically derive the equilibrium strategies in Budd, Harris, and Vickers [4] for the symmetric game with geometric effort cost functions. They then use this result to address the optimal design of the race. Moscarini and Smith take a different approach to solving the model, relying on the symmetry of MPE. Besides restricting their attention to symmetric equilibria, the authors consider only the case of no discounting. Another continuous time continuous state-space version of the Harris and Vickers model is developed in Horner [13]. He restricts the players to a binary effort choice. Hence, the MPE strategies are such that players switch between high and low effort only infrequently based on some threshold rule.

Even though in this paper I study a game with only two players, the method I developed here has been successfully applied to a model of team dynamics with many players in Georgiadis [7].

In the next section, I present the model and show the existence of MPEs with general cost functions under some weak restrictions, both with and without discounting. In Section 3, I study

[1] In the case with identical players, this closed form is the one found in Moscarini and Smith [15] up to an affine transformation.