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Stable games and their dynamics

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Abstract

We study a class of population games called *stable games*. These games are characterized by *self-defeating externalities*: when agents revise their strategies, the improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning. We prove that the set of Nash equilibria of a stable game is globally asymptotically stable under a wide range of evolutionary dynamics. Convergence results for stable games are not as general as those for potential games: in addition to monotonicity of the dynamics, integrability of the agents' revision protocols plays a key role.

JEL classification: C72; C73

Keywords: Population games; Evolutionarily stable strategies; Evolutionary dynamics; Global stability; Lyapunov functions

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1. Introduction

This paper studies a class of population games that we call *stable games*. These games are characterized by a condition we call *self-defeating externalities*, which requires that when agents revise their strategies, the improvements in the payoffs of strategies to which revising agents are switching are always exceeded by the improvements in the payoffs of strategies which revising agents are abandoning. Our main results show that the set of Nash equilibria of a stable game is globally asymptotically stable under a wide range of evolutionary dynamics, including the BNN dynamic, the best response dynamic, and the Smith dynamic. Related global stability results hold for the logit, replicator, and projection dynamics. But we argue that convergence results for stable games are not as general as those for potential games: in addition to monotonicity of the dynamics, integrability of the agents' revision protocols plays a key role.

Our treatment of stable games builds on ideas from a variety of fields. From the point of view of mathematical biology, one can view stable games as a generalization of the class of symmetric normal form games with an interior ESS (Maynard Smith and Price [37]) to settings with multiple populations and nonlinear payoffs. Indeed, for games with an interior Nash equilibrium, the ESS condition reduces to the negative definiteness of the payoff matrix, and this latter property characterizes the "strictly stable" games in this symmetric normal form setting.[1] Bishop and Cannings [5] shows that the war of attrition satisfies the weaker semidefiniteness condition that characterizes stable games. Stable single population games appear in the work of Akin [1], and stable multipopulation games with linear payoff functions are studied in Cressman et al. [10]. Stable games can be found in the transportation science literature in the work of Smith [54,55] and Dafermos [11], where they are used to extend the network congestion model of Beckmann et al. [3] to allow for asymmetric externalities between drivers on different routes. Alternatively, stable games can be understood as a class of games that preserves many attractive properties of concave potential games: in a sense to be made explicit soon, stable games preserve the concavity of these games without requiring the existence of a potential function at all. Stable games can also be viewed as examples of objects called monotone operators from the theory of variational inequalities.[2] Finally, as we explain in Section 2.4, stable games are related to the diagonally concave games introduced by Rosen [43].

To analyze the behavior of deterministic evolutionary dynamics in stable games, we first derive these dynamics from an explicit model of individual choice. This model is specified in terms of *revision protocols*, which determine the rates at which an agent who is considering a change in strategies opts to switch to his various alternatives. Most of our presentation focuses on a general class of dynamics called *target dynamics*, under which the rate at which an agent switches to any given strategy is independent of his current strategy choice. The name of this class of dynamics comes from their simple geometric description: for each current population state, the dynamics specify a *target state* toward which the current state moves, as well as a rate at which the target state is approached. To this we add the additional restriction that the rates at which agents switch to alternative strategies can be expressed as a function of the *excess payoff vector*: a vector whose ith component is the difference the ith strategy's payoff and the population's average payoff. Dynamics with these properties are prominent in the literature: three fundamental

[1] Conversely, symmetric normal form games with such a negative definite payoff matrix have a unique ESS which is also the unique Nash equilibrium; see [29].

[2] See [20,31,34,38,40].