Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 146 (2011) 1481–1499
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Minimal stable sets in tournaments

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Received 29 September 2009; final version received 8 January 2011; accepted 13 January 2011

Available online 19 May 2011

Abstract

We propose a systematic methodology for defining tournament solutions as extensions of maximality. The central concepts of this methodology are maximal qualified subsets and minimal stable sets. We thus obtain an infinite hierarchy of tournament solutions, encompassing the top cycle, the uncovered set, the Banks set, the minimal covering set, and the tournament equilibrium set. Moreover, the hierarchy includes a new tournament solution, the *minimal extending set*, which is conjectured to refine both the minimal covering set and the Banks set.

JEL classification: C71; C72; D71

Keywords: Tournament solutions; Stable sets; Social choice theory; Game theory

1. Introduction

Given a finite set of alternatives and choices between all pairs of alternatives, how to choose from the entire set in a way that is faithful to the pairwise comparisons? This simple, yet captivating, problem is studied in the literature on tournament solutions. A tournament solution thus seeks to identify the "best" elements according to some binary dominance relation, which is usually assumed to be asymmetric and complete. Since the ordinary notion of maximality may return no elements due to cyclical dominations, numerous alternative solution concepts have been devised and axiomatized (see, e.g., [22,20]). Tournament solutions have numerous applications throughout economics, most prominently in social choice theory where the dominance relation is typically defined via majority rule (e.g., [12,5]). Other application areas include multi-criteria

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0022-0531/$ – see front matter
doi:10.1016/j.jet.2011.05.004

decision analysis (e.g., [1,6]), non-cooperative game theory (e.g., [13,19,10]), and cooperative game theory [15,8].

In this paper, we approach the tournament choice problem using a methodology consisting of two layers: *qualified subsets* and *stable sets*. Our framework captures most known tournament solutions and allows us to provide unified proofs of axiomatic properties and inclusion relationships between tournament solutions.

In general, we consider six standard properties of tournament solutions: monotonicity (MON), independence of unchosen alternatives (IUA), the weak superset property (WSP), the strong superset property (SSP), composition-consistency (COM), and irregularity (IRR). The point of departure for our methodology is to collect the maximal elements of so-called qualified subsets, i.e., distinguished subsets that admit a maximal element. In general, families of qualified subsets are characterized by three properties (closure, independence, and fusion). Examples of families of qualified subsets are all subsets with at most two elements, all subsets that admit a maximal element, or all transitive subsets. Each family yields a corresponding tournament solution and we thus obtain an infinite hierarchy of tournament solutions. The tournament solutions corresponding to the three examples given above are the set of *all alternatives except the minimum*, the *uncovered set* [12,21], and the *Banks set* [2]. Our methodology allows us to easily establish a number of inclusion relationships between tournament solutions defined via qualified subsets (Proposition 2) and to prove that all such tournament solutions satisfy WSP and MON (Proposition 1). Based on an axiomatic characterization using minimality and a new property called strong retentiveness, we show that the Banks set is the finest tournament solution definable via qualified subsets (Theorem 1).

Generalizing an idea by Dutta [11], we then propose a method for refining any suitable solution concept S by defining minimal sets that satisfy a natural stability criterion with respect to S. A crucial property in this context is whether S always admits a *unique* minimal stable set. For tournament solutions defined via qualified subsets, we show that this is the case if and only if no tournament contains two disjoint stable sets (Lemma 2). As a consequence of this characterization and a theorem by Dutta [11], we show that an infinite number of tournament solutions (defined via qualified subsets) always admit a unique minimal stable set (Theorem 3). Moreover, we show that all tournament solutions defined as unique minimal stable sets satisfy WSP and IUA (Proposition 4), SSP and various other desirable properties if the original tournament solution is defined via qualified subsets (Theorem 4), and MON and COM if the original tournament solution satisfies these properties (Propositions 5 and 6). The minimal stable sets with respect to the three tournament solutions mentioned in the paragraph above are the *minimal dominant set*, better known as the *top cycle* [16,28], the *minimal covering set* [11], and a new tournament solution that we call the *minimal extending set (ME)*. Whether *ME* satisfies uniqueness turns out to be a highly non-trivial combinatorial problem and remains open. If true, *ME* would be contained in both the minimal covering set and the Banks set while satisfying all of the desirable properties listed above. We conclude the paper by axiomatically characterizing all tournament solutions definable via unique minimal stable sets (Proposition 7) and investigating the relationship between *ME* and the *tournament equilibrium set* [26].

When considering qualified subsets that are maximal in terms of cardinality rather than set inclusion and using a slightly modified definition of stability, our framework also captures quantitative tournament solutions such as the *Copeland set* and the *bipartisan set* (see [7]).