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Low T scaling in the binary $2d$ spin glass

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Abstract

We investigate $2d$ Ising spin glasses with binary couplings via exact computations of the partition function on lattices with periodic boundary conditions. After introducing the physical issues, we sketch the algorithm to compute the partition function as a polynomial with integer coefficients. This technique is then exploited to obtain the thermodynamic properties of the spin glass. We find an anomalous low temperature scaling of the heat capacity $c_v \sim e^{-2\beta}$ and that hyperscaling holds.

Keywords: Spin glasses; Critical phenomena; Partition function

1. The model

The Ising model (with ferromagnetic interactions J_{ij}=1 and its translational symmetry) can be seen as a first approximation of real systems where there are impurities. Considering disordered systems (more specifically, we shall deal with spin glasses) can be seen as a further step that increases the complexity: translational invariance is explicitly broken by the quenched disorder, and frustration appears through the competition of ferromagnetic and anti-ferromagnetic bonds:

$$H \equiv -\sum_{ij} J_{ij} S_i S_j. \quad (1)$$

Here the couplings J_{ij} are chosen from some symmetric distribution, for example a Gaussian distribution, or a bimodal distribution in which case $J_{ij}=\pm 1$; thus, the last case is referred to as the binary coupling case.

The typical approach of theoretical physics is to try to simplify situations that we know to be very complex, involving for example a huge number of degrees of freedom. One thus introduces models that can describe the macroscopic characteristics of the system without all the microscopic details. Then one wants to know whether one has a phase transition with the correct features, what is the nature of the critical behavior, what are the exponents that govern it...

A typical starting point is mean field theory. In the case of spin glasses, mean field theory has been solved [1] and it unveils a very complex structure in phase space (involving at low temperatures T, the so-called Replica Symmetry Breaking, RSB). What happens in finite dimensions remains controversial in spite of many years of work [2]. One expects the upper critical dimension to be 6, and the lower critical dimension to be close to 2.5: in three dimensions, we are very confident that there is a phase transition.

One wants to understand whether a spin glass phase (with a frozen state with no spontaneous magnetization) exists, and what happens as one gets close to the critical temperature T_c. One of the possible approaches to understand these systems is to compute directly what happens at T=0 by studying the ground state and excited states of the system.

Following the McMillan renormalization approach [3], the domain wall energy, that is the difference of ground-state

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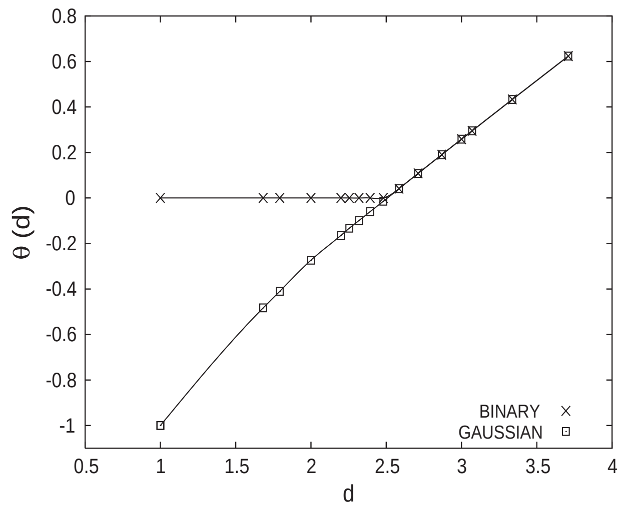

Fig. 1. The exponent θ in a family of Migdal–Kadanoff models defined for continuous dimensions d: note the difference between the Gaussian (□) and binary (x) models.

energies with periodic and anti-periodic boundary conditions, corresponds to the effective interaction of block spins at the scale L of the lattice size:

$$\Delta E = E_p - E_a \sim L^{\theta}, J \sim JL^{\theta}.$$

A first (slightly naive) classification says that depending on the sign of the exponent θ the system belongs to one of three different classes:

- $\theta>0$: the interaction becomes stronger at large distance, the thermal fluctuations are irrelevant and the frozen phase exists also at finite temperature;
- $\theta<0$: the thermal fluctuations destroy the spin glass order at any non-zero temperature;
- $\theta=0$: marginal behavior: the system is at its lower critical dimension, suggesting that the correlation length grows exponentially.

For the Ising spin glass in $2d$ with a Gaussian distribution of the disorder, the domain-wall exponent is negative ($\theta \approx -0.29$), and the spin glass phase does not survive at non-zero temperature. For the case of the bimodal distribution in $2d$, $\theta=0$. Following the classification we have given, this suggests that Ising spin glasses with two different realizations of the disorder (different microscopic characteristics) may belong to two different classes; then the microscopic characteristics would influence the critical behavior of the system.

This conclusion should hold if $\theta=0$ is really a complete signal of being at the lower critical dimension for the system. But is that true? If we look for example at a class of Migdal–Kadanoff approximations, for the case of the bimodal distribution, θ is equal to zero for all dimensions below or equal to the lower critical dimension (see Fig. 1); so, in this case, the fact that $\theta=0$ does not imply that one is at a lower critical dimension [4].

2. The heat capacity of the $\pm J$ spin glass

The best way to check what happens in our two dimensional spin glass is to look at the scaling of the correlation length close to the critical temperature. When $\theta<0$, the McMillan scaling picture shows that $\xi(T) \sim T^{1/\theta}$ as $T \rightarrow 0$. The thermal exponent ν is thus given by $\nu=-1/\theta$. This conclusion is fine (and is confirmed [11]) for the Ising spin glass with Gaussian couplings ($\theta \approx -0.29$). But in the case of the $\pm J$ spin glass, $\theta=0$, and so the expectation there is that ξ diverges faster than any inverse power of T as $T \rightarrow 0$. A likely behavior is that the correlation length grows exponentially in $1/T$.

Unfortunately looking directly at the correlation length can be cumbersome. Using Monte Carlo simulations, it is possible get all the information about configurations and energies, but the problem is equilibration at very low T (since $T_c=0$). A different possible approach is the calculation of the whole partition function: in this way, we can escape the problem of equilibration, but we do not have any information about the microscopic configurations of the elementary variables, and we have to rely on the measurement of some other thermodynamic quantity.

When calculating the partition function, we determine the degeneracy of all energy levels of the system: from that it is very easy extract the heat capacity c_v or other observables like the entropy. From these data we shall perform extrapolations to get the behavior in the thermodynamic limit.

The 1988 work by Wang and Swendsen [5] gives some information about the low T scaling of the heat capacity. Using an optimized Monte Carlo, they concluded that the scaling of c_v is "anomalous", in the sense that c_v scales for low T as $e^{-2\beta J}$, even though the energy gap is $4J$. Indeed, in standard systems, if $\Delta E = E_1 - E_0$ is the lowest excitation energy (the gap), c_v goes as $e^{-B\Delta E}$.

A possible explanation of this anomalous scaling may be found by looking the case of the one-dimensional (pure) Ising model. The pure Ising model has a marginal behavior in $1d$, its lower critical dimension. For the finite size system with **periodic boundary conditions** at fixed volume and very low T, c_v goes as $e^{-4\beta J}$, since the minimum energy excitation is $4J$ (see Fig. 2).

In Fig. 2 the lower lines are for spins $S_i=-1$, while the upper line is for spins $S_i=+1$. Starting from the ground state, if we want to flip spins and make a minimum energy excitation, because of the periodic boundary conditions, this minimum energy excitation has always an energy cost equal to $4J$ ($2J$ on the left, when the lower line jumps to the upper, and $2J$ on the right when the upper line jumps to the lower).

If we look to the $1d$ pure Ising model with free boundary conditions, the minimal energy excitation is $2J$, and

Fig. 2. A kink–antikink pair excitation when using periodic boundary conditions.