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Stronger measures of higher-order risk attitudes

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Abstract

This paper aims to extend the results by Ross (1981) [15] and by Modica and Scarsini (2005) [13] to stochastic dominance of degree 4 and over. Specifically, it is shown that Ross' approach can be extended to any order of risk attitude beyond the generalization proposed by Modica and Scarsini by means of sth degree increase in risk defined by Ekern (1980) [8].

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1. Introduction and motivation

In their founding paper, Arrow [1] and Pratt [14] defined the coefficient of absolute risk aversion $-u^{(2)}/u^{(1)}$, where $u^{(k)}$ stands for the kth derivative utility function u. This coefficient can then be used to compare the risk attitudes implied by two utility functions u and v. If, at all wealth levels, $-v^{(2)}/v^{(1)} \geqslant -u^{(2)}/u^{(1)}$ then the decision-maker with utility function v is more risk averse than the decision-maker with utility function u in the sense that he is always willing to pay more to get rid of any risk X.

Ross [15] showed that such a nice result does not extend to the case of a reduction in the risk X (instead of its elimination). In fact, when Y is a mean preserving contraction of X, the

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condition $-v^{(2)}/v^{(1)} \geqslant -u^{(2)}/u^{(1)}$ is no longer sufficient to guarantee that the decision-maker with utility function v is willing to pay more to benefit from the mean preserving contraction. Ross [15] then proposed a stronger definition of increased risk aversion that produces the desired intuitive result.

This line of research was pursued by Modica and Scarsini [13] who applied it to a situation of downside risk increase as defined by Menezes, Geiss and Tressler [12]. A by-product of their analysis is the idea that the coefficient of absolute downside risk aversion could be defined by $u^{(3)}/u^{(1)}$ instead of $-u^{(3)}/u^{(2)}$ as proposed by Kimball [10].

Our purpose in this paper is to show that Ross' approach can be extended to any order of risk attitude,[1] besides the third order one discussed by Modica and Scarsini [13]. This generalization is obtained by using some results about stochastic dominance that are recapitulated in Section 2. The next section contains the proof of our extension to any order of risk attitude. Before the conclusion, we show in the fourth section that our result suggests a candidate for the definition of the absolute index of nth order risk attitude.

2. *S*-increasing concave dominance rules

2.1. *S*-increasing concave utility functions

As decision-makers are usually assumed to be non-satiated and risk-averse, their utility function u is non-decreasing and concave. If u is differentiable, this means that the first derivative of u is non-negative and its second derivative is non-positive. More recently, it has been shown that higher-order derivatives of u also matter. Therefore, let us consider the non-decreasing utility functions with derivatives of degrees 1 to s of alternating signs. This property is satisfied by the utility functions most commonly used in mathematical economics including all the completely monotone utility functions such as the logarithmic, exponential and power utility functions. Formally, let us define the class $\mathcal{U}_{s\text{-icv}}$, $s = 1, 2, \ldots$, of the regular s-increasing concave functions as the class containing all the functions u such that $(-1)^{k+1}u^{(k)} \geqslant 0$ for $k = 1, \ldots, s$.

To get all the s-increasing concave utilities, we need to supplement $\mathcal{U}_{s\text{-icv}}$ with all the pointwise limits of elements in $\mathcal{U}_{s\text{-icv}}$. This gives the class $\overline{\mathcal{U}}_{s\text{-icv}}$ of all the utilities such that $(-1)^{k+1}u^{(k)} \geqslant 0$ for $k = 1, \ldots, s-2$ and $(-1)^{s-2}u^{(s-2)}$ is non-decreasing and concave.

Letting s tend to $+\infty$ gives utilities with all odd derivatives positive and all even derivatives negative. In this case, utility functions are completely monotone and express mixed risk aversion, as studied in Caballé and Pomansky [2].

2.2. *Forward differences, lotteries, and allocation of additional resources*

Let us now give a precise economic meaning for the sign of the kth derivative of the utility function u. This is done by defining sequences of lotteries, similar to those used by Caballé and Pomansky [2] to characterize mixed risk aversion (see Proposition 3.1 in their paper). See also Denuit, Lefèvre and Scarsini [4]. This explanation complements the analysis conducted first by Eeckhoudt and Schlesinger [6] through the concept of risk apportionment and later by

[1] A similar result is obtained by Jindapon and Neilson [9] but they follow a very different route. While their result is based on an original comparative statics analysis, ours is developed in terms of preferences in a context-free framework as in Ross [15]. Notice besides that Li [11] also independently investigated a context-free higher-order extension of Ross [15] using the approach of Jindapon and Neilson [9].