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# Linear-price term structure models

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ABSTRACT

We characterize the term structure models in which the zero-coupon prices are linear functions of underlying factors. These models are called Linear-price Term Structure Models (LTSM). We provide two types of LTSM where the observable factors predict regimes which are not observed by the investor. These hidden regimes are represented by a Markov chain, which features either an exogenous, or an endogenous dynamics. We illustrate the possible term structure patterns, their evolutions, in particular their ability to stay close to a zero lower bound.



## 1. Introduction

The dynamic analysis of the term structure of interest rates is generally based on Affine (yield) Term Structure Models (ATSM), in which the interest rates at different time-to-maturity are affine functions of a finite (or infinite) set of factors. The ATSM includes a number of well-known term structure models such as the Vasicek and Cox, Ingersoll, Ross single factor models, the Langetieg, Quadratic and Wishart multifactor models [see e.g. Dai and Singleton (2000), Ahn et al. (2002), Gourieroux (2006), Gourieroux et al. (2010)]. The ATSM have shown their flexibility to provide various term structure patterns or to capture the effects of default risk [see e.g. Duffie and Singleton (1999), Gourieroux et al. (2006), Monfort and Pegoraro (2007)].

However, there exist alternative flexible specifications. As noted in Siegel (2010), it is a "relativity recent idea to represent bond prices, instead of yields, using a linear factor model". Thus we get a Linear-Price Term Structure Model (LTSM)[1]. This new class of term structure models has both interesting technical and practical features. For instance, the no arbitrage restrictions are easy to derive, and the LTSM may be more appropriate than the ATSM to account for regimes hidden to the investors, or for representing special features of interest rates, such as the short rates close to zero recently observed [see Ichine and Ueno (2007), Kim and Singleton (2010)]. Moreover we consider two important classes of LTSM: the Hidden Markov Term Structure Models (HMTSM) and the Finite Dependence Term Structure Models (FDDTSM), which provide quasi explicit formulas for bond prices and European derivatives and which are also easily tractable from an econometric point of view.

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[1] We have followed the standard terminology of (Linear) Affine Term Structure Model, when the yields are (linear) affine functions of factors. However, the terminology Linear Term Structure Model might appear in the literature for Linear Affine Term Structure Model, when the word Affine is unfortunately omitted. Such an omission appears e.g. in Cochrane (2001), in the title of Section 19.5, which has to be read as "Three Linear Affine Term Structure Models", as immediately seen from the content.

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The paper is organized as follows. We first introduce in Section 2, term structure models in which the prices of zero coupon bonds are linear combinations of a finite number of stochastic dynamic factors. We discuss the restrictions on the term structure pattern implied by no arbitrage. Then, we describe two types of LTSM. The Hidden Markov Term Structure Model (HMTSM) is introduced in Section 3. The class is constructed by introducing an exogenous Markov chain which is not observed by the investors, and observed variables, whose evolution depends on the latent exogenous Markov chain. We show that this construction leads to LTSM in bond prices, and we explain how to derive the associated factors. These factors have a complicated nonlinear dynamic with a long memory. We explain how the pricing formula could be extended to any type of interest rate derivatives, leading also to a derivative price, which is linear in the same observable factors. In Section 4 another LTSM is introduced and discussed. This model is based on the notion of Markov process with finite dimensional dependence. We derive the expression of the observable factors. These models are called the Finite Dimensional Dependence Term Structure Models (FDDTSM). In some special cases, these models can be interpreted in terms of hidden Markov chain, where the unobservable Markov chain is now endogenous. An illustration of the two types of term structure models is provided in Section 5. We compare the ability of both models to capture the stylized facts observed when there is a zero binding floor for the short-term rate. We show that the FDDTSM is more flexible for this purpose. Section 6 concludes. The proofs are gathered in.

## 2. Linear-Price Term Structure Models

Let us now focus on Linear-Price Term Structure Models (LTSM) defined by:

$$B(t,H) = a(H)'F_t, \forall H, 1 \le H \le \overline{H}, \tag{2.1}$$

where $B(t, H)$ is the price at time $t$ of a zero-coupon bond with residual maturity $H$ and $F_t$ is a set of $K$ linearly independent stochastic factors. One of these factors can possibly be constant in time, and in this case the linear combination includes an intercept.

### 2.1. The term structure pattern

The factor coefficients $a(H)$ cannot be chosen independently, but may be subject to no arbitrage restrictions. We assume below:

**Assumption A.1.** The information available to the investors at date $t$ includes the current and lagged values of the factors (and also due to Eq. (2.1) the current and lagged values of the prices of the zero-coupon bonds).

**Assumption A.2.** The markets of zero-coupon bonds are liquid, that is, any quantity of a zero-coupon bond with maturity $H$ can be traded at date $t$ at the unitary price $B(t, H)$, for any time-to-maturity $H, 1 \le H \le \overline{H}$.

**Assumption A.3.** There is no short-sell restriction.

Under Assumptions A.2–A.3, the investor can trade any coupon bond, with possibly negative coupon for some time-to-maturity. Moreover, under no arbitrage opportunity, the price of such a coupon bond is equal to the combination of the prices of the zero-coupon bonds with the coupons as coefficients. Then, under Eq. (2.1), the prices of the coupon bonds form a vector space, which is included in the space generated by the components of $F_t$.

Let us now explicit the restrictions on coefficients $a(H)$ due to the absence of arbitrage opportunity. We have the following result:

**Proposition 1.** *In a Linear-Price Term Structure Model (2.1) and under Assumptions A1–A3, the absence of arbitrage opportunity for a self-financed portfolio of zero-coupon bonds implies that there exists a matrix C such that*:

$$a(H)' = a(1)'C^{H-1}. \tag{2.2}$$

**Proof.** See Appendix 1.

This type of restriction does not depend on the maximal time-to-maturity $\overline{H}$ easily tradable on the market. Under conditions (2.2), the $K\overline{H}$ factor coefficients depend at most on $K + K^2$ underlying parameters, which are the elements of $a(1)$ and matrix $C$, respectively. Some of these coefficients, namely $a(1)$, are characterizing the short-term rate, whereas matrix $C$ characterizes the pattern of the term structure of zero-coupon prices.

### 2.2. The multiplicity of factor representations

i) The expression of the discount function, that is the sequence of zero-coupon prices with respect to time-to-maturity, implied by Proposition 1 can be written in an equivalent way since the factors are defined up to a one-to-one linear transform. More precisely we can write:

$$B(t,H) = a(1)'M^{-1}\left(MCM^{-1}\right)^{H-1}MF_t,$$