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Proposing indicators to measure achievement and shortfall inequality consistently☆

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ABSTRACT

In several economic fields, such as those related to health or education, the individuals' characteristics are measured by bounded variables. Accordingly, these characteristics may be indistinctly represented by achievements or shortfalls. A difficulty arises when inequality needs to be assessed. One may focus either on achievements or on shortfalls but the respective inequality rankings may lead to contradictory results. In this note we propose a procedure to define indicators that measure equally the achievement and shortfall inequality. Specifically, we derive measures which are invariant under ratio-scale or translation transformations, and a decomposable measure is also obtained. As the indicators proposed depend on the distribution bounds, families of indices that guarantee the same inequality rankings regardless of the distribution maximal levels are identified.

1. Introduction

A number of recent papers have highlighted the difficulties in measuring inequality of a distribution that can be described either in terms of achievements or shortfalls (among them Clarke et al., 2002; Erreygers, 2009c; Lambert and Zheng, 2011). This situation arises in different economic fields in which bounded variables are involved, particularly in the measurement of health inequality. As stressed in the mentioned papers, the choice between achievement and shortfall inequality measurement is not innocuous, since different choices may lead to contradictory results.

Erreygers (2009c) characterizes two indicators, appropriate normalizations of the absolute Gini index and the coefficient of variation, respectively, both depending on the distribution bounds, which measure achievement and shortfall inequality identically. The square of the latter is decomposable in the sense that the overall inequality can be expressed as a weighted sum of the inequality levels computed for population subgroups plus inequality arising from the differences among subgroup means. In turn, Lambert and Zheng (2011) introduce a weaker property to measure achievement and shortfall inequality consistently, and show that all relative and intermediate standard inequality indices fail their requirement. They also identify two classes of absolute inequality indices according to which the measure of achievement and shortfall inequality is identical, and show that, among them only the variance is subgroup decomposable.

All these results rightly consider that achievements and shortfalls are different sides of the same coin and, consequently, inequality of shortfalls and inequality of achievements should mirror each other. Our starting point is slightly different. In fact, this paper proposes considering a unified framework where the achievement and the shortfall distributions can be jointly analyzed. One simple way to do this, given any inequality measure, is to aggregate the respective achievement and shortfall inequality levels in a single indicator. Section 3 shows that taking a generalized mean of these two values allows us to transform any inequality measure into an indicator which is able to capture the achievement and the shortfall inequality consistently. In addition, some

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of the properties enjoyed by the original index are inherited by its transformation. Accordingly, measures both ratio-scale and translation invariant, may be obtained and a decomposable index is also identified.

When a standard inequality index is used to measure shortfall inequality, the results depend on the bounds of the distribution. The same happens if the indicators we suggest are applied. Most times these levels are fixed values, for instance, if attainment is measured by a variable in percentage terms. However, it may be the case that the bounds change. Then the procedure proposed will introduce a source of arbitrariness in the measurement since inequality orderings may change when the bounds vary. Hence, Section 4 is devoted to obtaining inequality indicators that are bound-consistent, that is, they lead to the same orderings regardless of the bounds. A family of decomposable indices which gauges shortfall inequality bound-consistently is characterized. We also identify indices for which the geometric mean aggregator rankings are independent of the bounds. Finally we show that, in a decomposable setting, only absolute measures can be aggregated through the arithmetic mean indicator so that the inequality orderings remain unchanged when the bounds vary.

2. Notation and basic definitions

We consider a population consisting of $n \geq 2$ individuals. An *achievement distribution* is represented by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in D^n$, with $D^n = \mathbb{R}^n_+$ or $D^n = \mathbb{R}^n_{++}$, where x_i represents individual i's achievement. We assume that the variables are ratio-scale and are lower bounded by 0. The set of all feasible distributions is $D = \bigcup_{n \geq 2} D^n$. The positive part will be denoted by D_+. For any $\mathbf{x} \in D$, $\mu_{\boldsymbol{x}} = \mu(\boldsymbol{x})$ and $n_{\boldsymbol{x}} = n(\boldsymbol{x})$ stand, respectively, for the mean and population size of the distribution $\mathbf{x}$.

For each $\alpha > 0$ we let D^α represent the set of distributions for which α is an upper bound and denote as $D^\alpha_+ = \{\boldsymbol{x} \in D_+ / x_i < \alpha\}$. Note that if $\alpha' > \alpha$, then $D^{\alpha'} \supset D^\alpha$ and D can be decomposed as $D = \bigcup_{\alpha > 0} D^\alpha$. The *shortfall distribution* associated with $\boldsymbol{x} \in D^\alpha$ is denoted as $\boldsymbol{s} = (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n_+$, where $s_i = \alpha - x_i$ represents individual i's shortfall. We use the notation $\mathbf{1} = (1, \ldots, 1)$ and $\lambda\mathbf{1} = (\lambda, \ldots, \lambda)$. Hence the shortfall distribution can be equivalently denoted by $\boldsymbol{s} = \alpha\mathbf{1} - \boldsymbol{x}$.

Given two distributions $\boldsymbol{x}, \boldsymbol{x}' \in D$, we say that $\boldsymbol{x}'$ is obtained from $\boldsymbol{x}$ by a *progressive transfer* if there exist two individuals $i, j \in \{1, \ldots, n\}$ and $h > 0$ such that $x'_i = x_i + h \leq x_j - h = x'_j$ and $y'_k = y_k$ for every $k \neq i, j$.

An inequality index I is a real valued continuous function $I : D \to \mathbb{R}$ which fulfils the following properties.

Pigou-Dalton Transfer Principle (TP). $I(\boldsymbol{x}') < I(\boldsymbol{x})$ whenever $\boldsymbol{x}'$ is obtained from $\boldsymbol{x}$ by a progressive transfer.

Normalization (NOR). $I(\lambda\mathbf{1}) = 0$ for all $\lambda > 0$.

Symmetry (SYM). $I(\boldsymbol{x}) = I(\boldsymbol{x}')$ whenever $\boldsymbol{x} = \Pi\boldsymbol{x}'$ for some permutation matrix Π.

Replication Invariance (RI). $I(\boldsymbol{x}) = I(\boldsymbol{x}')$ whenever $\boldsymbol{x}' = (\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x})$ with $n_{\boldsymbol{x}'} = mn_{\boldsymbol{x}}$ for some positive integer m.

The crucial axiom in inequality measurement is the *Pigou-Dalton transfer principle* which requires that a transfer from a richer person to a poorer one decreases inequality. In addition, the indices are usually assumed to be *normalized* with the inequality level equal to 0 when everybody has exactly the same distribution value. *Symmetry* establishes that the inequality index should be insensitive to a reordering of the individuals. Finally, *replication invariance* allows populations of different sizes to be compared. These four properties are considered to be inherent to the concept of inequality and have come to be accepted as basic properties for an inequality index.

An inequality index $I_R : D_+ \to \mathbb{R}$ is *relative* if proportional changes in all the values do not alter the inequality level, that is, for all $\mathbf{x} \in D_+$ $I(\lambda x) = I(x)$ where $\lambda > 0$.

A relative index is insensitive to variations in the unit in which the variables are measured.

An inequality index $I_A : D \to \mathbb{R}$ is *absolute* if the same increase in all the distribution values does not change the inequality level, that is, for all $\boldsymbol{x} \in D$ $I(\boldsymbol{x} + \eta\mathbf{1}) = I(\boldsymbol{x})$ for all η whenever $\boldsymbol{x} + \eta\mathbf{1} \in D$.

Given an inequality measure I and $\alpha > 0$, $I^S(.;\alpha)$ stands for the *shortfall indicator* defined as $I^S(\boldsymbol{x};\alpha) = I(\alpha\mathbf{1} - \boldsymbol{x})$ for any $\boldsymbol{x} \in D^\alpha$.

3. Proposing perfect complementary indicators.

3.1. The r-indicators associated with an inequality measure.

This paper deals with the problem of evaluating and comparing the inequality level of bounded distributions. In these cases, a person's characteristics can be represented in terms of achievements or in terms of shortfalls. Consequently, the inequality level can be assessed focusing on either of these terms. These two frameworks are linked but nevertheless distinct, and can yield different results. As mentioned above, recent efforts have been made to introduce conditions and to define indicators for which the respective inequality levels mirror each other.

This paper aims to propose a mixed approach in which achievements and shortfalls may be jointly analyzed. We may think of the inequality of a bounded distribution as an aggregate of the inequality of achievements and the inequality of shortfalls. The properties enjoyed by the r-order means make them an appropriate way of aggregation in several economic fields. As will be showed, also in this framework they behave in a satisfactory way.

Consider an inequality measure I, a maximum level of achievements α, and, for a given distribution $\boldsymbol{x} \in D^\alpha$, the inequality values $I(\boldsymbol{x})$ and $I(\alpha\mathbf{1} - \boldsymbol{x})$. If we are interested in analysing simultaneously the achievement and the shortfall inequality, we may think of aggregating these two values. A natural aggregation procedure may be any r-order mean of them. The indicator defined in such a way depends on the distribution $\mathbf{x}$ and on bound α.

Specifically, given $\alpha > 0$ we propose to consider the *r-indicator associated with I*, denoted by I^r that, for each distribution $\boldsymbol{x} \in D^\alpha$, takes the following value

$$I^r(\boldsymbol{x};\alpha) = \begin{cases} \left(\dfrac{I(\boldsymbol{x})^r + I(\alpha\mathbf{1} - \boldsymbol{x})^r}{2}\right)^{1/r} & \text{if } r \neq 0 \\ (I(\boldsymbol{x})I(\alpha\mathbf{1} - \boldsymbol{x}))^{1/2} & \text{if } r = 0 \end{cases}$$

When $r < 0$, the r-order means are defined only for positive values. However, as $I(x) = 0$ implies, by normalization, that $\boldsymbol{x} = k\mathbf{1}$, we will take the convention that whenever $I(\boldsymbol{x}) = I(\alpha\mathbf{1} - \boldsymbol{x}) = 0, I^r(\boldsymbol{x};\alpha) = 0$ for any $r < 0$.

Now some properties of the r-order means are mentioned. For any r, $I^r(\boldsymbol{x};\alpha)$ lies between $I(\boldsymbol{x})$ and $I(\alpha\mathbf{1} - \boldsymbol{x})$. Particular members of this family are $I^1(\boldsymbol{x};\alpha)$, which corresponds to the arithmetic mean of the two values and $I^0(\boldsymbol{x};\alpha)$, the geometric mean. The mapping $r \to I^r(\boldsymbol{x};\alpha)$ is a non decreasing continuous function on all of $\mathbb{R}$. The limiting case at one extreme is as $r \to -\infty$, giving $I^r(\boldsymbol{x};\alpha) \to \min\{I(\boldsymbol{x}), I(\alpha\mathbf{1} - \boldsymbol{x})\}$. At the other extreme, as I^r, giving $I^r(\boldsymbol{x};\alpha) \to \max\{I(\boldsymbol{x}), I(\alpha\mathbf{1} - \boldsymbol{x})\}$. Moreover, for a given r, $I^r(\boldsymbol{x};\alpha)$ is non-decreasing in $I(\boldsymbol{x})$ and in $I(\alpha\mathbf{1} - \boldsymbol{x})$.[2]

In what follows we show that some additional properties fulfilled by I are inherited by the r-indicators.

[2] See for example Steele (2004).