Available online at www.sciencedirect.com
SCIENCE DIRECT®
International Journal of Solids and Structures 42 (2005) 2181–2197
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Dynamic stiffness formulation and free vibration analysis of a three-layered sandwich beam

J.R. Banerjee *, A.J. Sobey

School of Engineering and Mathematical Sciences, City University, Northampton Square, London EC1V OHB, UK

Received 16 July 2004; received in revised form 10 September 2004
Available online 5 November 2004

Abstract

A dynamic stiffness theory of a three-layered sandwich beam is developed and subsequently used to investigate its free vibration characteristics. This is based on an imposed displacement field so that the top and bottom layers behave like Rayleigh beams, whilst the central layer behaves like a Timoshenko beam. Using Hamilton's principle the governing differential equations of motion of the sandwich beam are derived for the general case when the properties of each layer are dissimilar. For harmonic oscillation the solutions of these equations are found in exact analytical form, taking full advantage of the application of symbolic computation, which has also been used to obtain the amplitudes of axial force, shear force and bending moment in explicit analytical forms. The boundary conditions for responses and loads at both ends of the freely vibrating sandwich beam are then imposed to formulate the dynamic stiffness matrix, which relates harmonically varying loads to harmonically varying responses at the ends. Using the Wittrick–Williams algorithm the natural frequencies and mode shapes of some representative problems are obtained and discussed. The important degenerate case of a symmetric sandwich beam is also investigated.

Keywords: Dynamic stiffness method; Free vibration; Sandwich beam; Wittrick–Williams algorithm; Symbolic computation

1. Introduction

Sandwich construction offers the structural designer many attractive features, such as high specific stiffness, good buckling resistance, formability into complex shapes, easy reparability, and so on. Thus the analysis of such structural systems has been investigated–more or less continuously–for well over half a century. There are some excellent papers which contribute to the state-of-the-art, review earlier work and

* Corresponding author. Tel.: +44(0)2070408924; fax: +44(0)2070408566.
E-mail address: j.r.banerjee@city.ac.uk (J.R. Banerjee).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.09.013

provide a long list of references on the subject, see for example, Frostig and Baruch (1994), Silverman (1995), Sainsbury and Zhang (1999) and Austin and Inman (2000).

By introducing visco-elasticity into the central element, good energy dissipation may be realised, but this paper does not deal with such non-conservative systems, but examines, in a way that has not been previously presented, the dynamical behaviour of an asymmetric three-layered element, namely a beam with three distinct components.

DiTaranto (1965), Mead and Sivakumaran (1966), Mead and Marcus (1969) and Mead (1982) are some of the earlier investigators who have examined the free vibration problems of sandwich beams using analytical methods, with due attention to the shearing that occurs between layers. Any Newtonian approach must take these shearing loads into account, with consequent complications in the analysis. Several other authors (Sakiyama et al., 1996; Fasana and Marchesiello, 2001; Banerjee, 2003) have also addressed this problem using an analytical model in which the top and bottom elements behave like beams in Bernoulli–Euler flexure, with the central element deforming only in transverse shear. Further developments that extend this model by adding a direct stress carrying capability to the central element can be found in He and Rao (1993), Bhimaraddi (1995) and Sisemore and Darvennes (2002). In many of these earlier works, assumptions such as the congruence of the top and bottom layers seriously restrict the value of the models.

The problems of modelling shears interacting between the layers can largely be side stepped by using an energy model, and thus implicitly, but not explicitly, representing the interaction. In particular, the use of Hamilton's principle allows a model to be developed in which the best possible elastic representation is achieved, subject to whatever restrictions are built into the analytical model. For example, a displacement field may be imposed which allows each element to behave in a relevant manner and as long as the representation can be justified, a good model will be realised.

This paper develops a model along these lines for the three-layered beam with no restrictions on the geometric and physical properties of each element. The three elements all have, in general, a mean axial (or longitudinal) motion as well as a common flexure and the system is fully coupled so that when the beam flexes, it has longitudinal response and vice versa. This model is of eighth order, and therein lies the difficulty in analytical development.

As is shown below, the completion of the analysis is achieved by using symbolic computation (Fitch, 1985; Rayna, 1986). This makes possible the development of a model in which the only approximations introduced are in the choice of the displacement field. The end product of the investigation is the development, and application, of the dynamic stiffness matrix of the three-layered beam. This retains all information derived by solving the governing differential equations subject to the appropriate boundary conditions. Thus the only source of error is in the choice of the displacement field.

The system of displacements used is as follows. All three layers have a common flexure. The top and bottom layers of the beam are assumed to bend in such a way that the cross-section rotates so as to be normal to the mid-plane flexure, as in the case of a Bernoulli–Euler beam, but with a longitudinal displacement and rotatory inertia taken into account. Thus the axial displacement varies linearly through the thickness. The central element also has a linear variation in axial displacement, but the cross-section does not rotate so as to be normal to the common flexure, and necessarily shears. This is modelled as a Timoshenko beam.

Such a procedure cannot generate a complete solution to the boundary value problem of the beam in vibration because it does not allow for variation in the transverse shear (and any associated non-planar bending). However, the likelihood is that the boundary zone between layers in which the shear changes rapidly is quite thin and so the inherent inaccuracy in the displacement field introduces only a small error into the energy expressions formulated, and so does not degrade significantly the whole model.

The resulting set of differential equations, which governs the free vibration of the three-layered beam, is of eighth order, which only degenerates into a simpler system of a sixth order flexure and second order longitudinal motion in the exceptional case when the top and bottom layers are identical. This is proved below.