ELSEVIER

Journal of Mathematical Behavior 24 (2005) 351–360

Mathematical Behavior

www.elsevier.com/locate/jmathb

Problem-solving, proving, and learning: The relationship between problem-solving processes and learning opportunities in the activity of proof construction

Keith Weber

Graduate School of Education, Rutgers University, 10 Seminary Place, New Brunswick, NJ 08901, USA

Available online 30 September 2005

Abstract

In university mathematics courses, the activity of proof construction can be viewed as a problem-solving task in which the prover is asked to form a logical justification demonstrating that a given statement must be true. The purposes of this paper are to describe some of the different types of reasoning and problem-solving processes used by undergraduates to construct proofs in their university mathematics courses, and to consider the relationship between the reasoning that students use when constructing a proof and what they have the opportunity to learn from their proving experience.

Keywords: Learning; Problem solving; Problem-solving processes; Proof; Reasoning

1. Introduction

1.1. Proving as a problem-solving task

Proving is a complex mathematical activity with logical, conceptual, social, and problem-solving dimensions. Consequently, researchers in mathematics education can fruitfully investigate pedagogical issues associated with proof from a variety of perspectives. Many researchers have emphasized students' conceptions of proof, illustrating how students' beliefs about what constitutes a convincing mathematical argument can influence their proof-related behavior (e.g., Harel & Sowder, 1998). Other researchers have focused on the social nature of proof and strive to create classroom environments in which students negotiate standards for what constitutes an acceptable mathematical justification (e.g., Alibert and Thomas, 1991). Still others have treated proof as a formal construct and examined the logical skills that are necessary for proving competence (e.g., Selden & Selden, 1995).

In this paper, I will consider proof from an alternative perspective, viewing proof construction as a problem-solving task. I first define a mathematical task as a situation in which an individual is presented with an initial set of information and is asked to derive a piece of desired information through the application of permissible mathematical actions and operations (cf. Anderson, 2000). An exercise is a mathematical task in which it is obvious to the individual which mathematical actions should be applied. A mathematical problem is a task in which it is not clear to the

E-mail address: khweber@rci.rutgers.edu.

0732-3123/$ – see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmathb.2005.09.005

individual which mathematical actions should be applied, either because the situation does not immediately bring to mind the appropriate mathematical action(s) required to complete the task or because there are several plausible mathematical actions that the individual believes could be useful. Proof construction is a mathematical task in which the prover is provided with some initial information (e.g., assumptions, axioms, definitions) and is asked to apply rules of inferences (e.g., recall previously established facts, apply theorems) until a desired conclusion is deduced. Further, proof construction is usually a problem-solving task. In most proving situations, there are dozens of valid inferences that one could draw, but only a small number of these inferences will be useful in constructing a proof (Weber, 2001).

Researchers in mathematics education generally have not taken the perspective of proving as problem solving; most researchers investigating proof instead choose to focus on students' beliefs and other affective issues (Mamona-Downs & Downs, this issue). Clearly, research on these issues is important as students are deeply perplexed by proof and their confusion has a significant impact on their behavior when constructing proofs. However, focusing on the problem-solving aspects of proving allows insight into some important themes that other perspectives on proving do not address, including the heuristics that mathematicians use to construct proofs (Polya, 1945), reasons that students reach impasses in proof where they do not know how to proceed (Schoenfeld, 1985; Weber, 2001), and techniques for teaching students strategies and heuristics to overcome their strategic difficulties associated with proving (e.g., Koedinger & Anderson, 1993; Weber, in press-a).

1.2. The relationship between problem-solving, proving and learning

In the past three decades, there has been a dramatic shift in the nature of research on mathematical problem-solving. In the 1980s, problem-solving was primarily conceived of as a goal of mathematical instruction. Research was undertaken with the aim of understanding the nature of problem-solving and creating instructional programs that developed students' knowledge base, heuristics, and dispositions so that they could solve problems more effectively (Schoenfeld, 1992). In recent years, it has become more common to view problem-solving as a means to achieve other pedagogical goals (Stacey, this issue). Problem-solving situations can be used as valuable pedagogical tools for helping students construct sophisticated mathematical knowledge (Schroeder & Lester, 1990). Under the right circumstances, engaging students in problem-solving can foster the development of deep mathematical insight, useful representations for reasoning about complex mathematical concepts, and powerful problem-solving heuristics (cf. Maher, 2002).

Although presenting students with interesting problems can positively contribute to their mathematical growth, this does not always occur. When analyzing what an individual learns from a problem-solving episode, it is insufficient to consider only the problem that the individual has attempted and the solution that he or she obtained. One would also need to take into account the processes that the individual used to obtain that solution (Lithner, 2003; Nunokawa, this issue). If students use a shallow strategy for solving a problem, such as copying the solution of a similar problem from a textbook and changing a few variables, then the learning obtained from this episode is likely to be rather limited. On the other hand, if students use what Lithner calls "plausible reasoning" by basing reasoning on the intrinsic properties of relevant mathematical concepts, then substantial learning can occur (Lithner, 2003). Nunokawa (this issue) further investigates the relationship between problem-solving and learning. He illustrates how and what a student learns from solving a problem is critically dependent upon the way the problem is solved, what details of the problem are attended to, and the scaffolding that the teacher provides.

Previous research on proof with a problem-solving perspective has often viewed proving as a primary goal of instruction (cf. Koedinger & Anderson, 1993; Weber, 2001). However, many researchers in mathematics education also view proving as a learning tool to gain conviction that assertions are true (e.g., Rodd, 2000), understand why assertions are true (e.g., Hanna, 1990; Nunokawa, this issue), and develop or refine one's representations and intuitions about mathematical concepts (e.g., Pinto & Tall, 1999). Like most problem-solving tasks, there is no one "right" way to complete the task of proof construction. Recent research has demonstrated that an individual can successfully construct proofs in a variety of qualitatively different ways (e.g., Alcock & Weber, in press; Pinto & Tall, 1999; Raman, 2003; Weber & Alcock, 2004). The central argument that I will present in this paper is that what one learns from the activity of proving does not only depend on the theorem that one is proving or the proof that one produces. The approach that an individual takes to constructing a proof will influence what learning opportunities are afforded by the proof production.