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Fuzzy bi-ideals in ordered semigroups

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Dedicated to Professor Lotfi A. Zadeh

Abstract

Given a set S, a fuzzy subset of S is, by definition, an arbitrary mapping $f : S \to [0, 1]$ where $[0, 1]$ is the unit segment of the real line. If the set S bears some structure, one may distinguish some fuzzy subsets of S in terms of that additional structure. This important concept was first introduced by Zadeh. Fuzzy groups have been first considered by Rosenfelt, fuzzy semigroups by Kuroki. A theory of fuzzy sets on ordered groupoids and ordered semigroups can be developed. In the present paper we endow S with the structure of an ordered semigroup and define "fuzzy" analogous for several notions that have been proved to be useful in the theory of ordered semigroups. We define the fuzzy bi-ideals in ordered semigroups and we give the main theorem which characterizes the bi-ideals in terms of fuzzy bi-ideals. Then we characterize the left and right simple, the completely regular, and the strongly regular ordered semigroups by means of fuzzy bi-ideals. We also study the decomposition of left and right simple ordered semigroups and of ordered semigroups having the property $a \leqslant a^2$ for all a, in terms of fuzzy bi-ideals. This decomposition is uniquely defined.

Keywords: Fuzzy subsemigroup; Fuzzy bi-ideal in an ordered semigroup; Left (right) ideal; Ideal; Filter; Semiprime elements in ordered semigroups; Congruence; Semilattice congruence; Complete semilattice congruences on an ordered semigroup; Regular; Left (right) simple; Completely regular; Duo; Strongly regular ordered semigroups; Semilattices of left and right simple semigroups in ordered semigroups

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1. Introduction

Given a set S, a fuzzy subset of S (or a fuzzy set in S) is, by definition, an arbitrary mapping $f : S \to [0,1]$ where $[0,1]$ is the usual interval of real numbers. This important concept of a fuzzy set has been introduced by Zadeh in [1]. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, groupoids, real analysis, measure theory, topology, etc. Rosenfeld [2] was arguably the first who considered the case when S is a groupoid. He gave the definition of a fuzzy subgroupoid and the fuzzy left (right, two-sided) ideal of S and justified these definitions by showing that a (conventional) subset A of a groupoid S is a (conventional) subgroupoid or a left (right, two-sided) ideal of S if the characteristic function

$$f_A : S \to [0,1]$$

defined by

$$x \to f_A(x) := \begin{cases} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A \end{cases}$$

is, respectively, a fuzzy subgroupoid or a fuzzy left (right, two-sided) ideal of S.

See also Kuroki's papers [3–8] who studied the fuzzy sets on semigroups and the paper by Liu [9] where "fuzzy" analogous of several further important notions, e.g. those of a bi-ideal or an interior ideal have been defined and justified in a similar fashion. Characterizations of groups, union of groups and semilattices of groups by means of fuzzy bi-ideals have been considered by Kuroki in [3]. Fuzzy sets on semigroups have been also considered by Kehayopulu, Xie and Tsingelis in [10] and by Kehayopulu and Tsingelis in [11–14]. A theory of fuzzy sets on ordered groupoids–semigroups can be developed. Following the terminology given by Zadeh, if S is an ordered groupoid (resp. ordered semigroup), a fuzzy set in S (or a fuzzy subset of S) is any mapping of S into the real closed interval $[0,1]$. Based on the terminology given by Zadeh, fuzzy sets in ordered groupoids have been first considered by Kehayopulu and Tsingelis in [15]. They then defined "fuzzy" analogous for several notions that have been proved to be useful in the theory of ordered groupoids–semigroups. Moreover, each ordered groupoid can be embedded into an ordered groupoid having a greatest element (poe-groupoid) in terms of fuzzy sets [16]. The concept of a bi-ideal B of a semigroup S has been first introduced by Good and Hughes in [17] as a subsemigroup of S having the property $BSB \subseteq B$ and studied in several items by S. Lajos (cf. also [18, pp. 84–85]). Keeping in mind the terminology of bi-ideals of semigroups mentioned above, we extend it to ordered semigroups by defining a bi-ideal of an ordered semigroup S as a subsemigroup of S having the properties: (1) $BSB \subseteq B$ and (2) If $a \in B$ and $S \ni b \leqslant a$, then $b \in S$.

In this paper, we first give the main theorem which characterizes the bi-ideals of ordered semigroups in terms of fuzzy bi-ideals. In this respect, we