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Likelihood relations and stochastic preferences[☆]

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ABSTRACT

We define the concept of a qualitative, non-numerical relative likelihood relation, to capture the intuition that "it is at least as likely that *a* is preferred to *b*, as that *c* is preferred to *d*." We provide necessary and sufficient conditions for this concept to be a basis for the numerical concept of a stochastic preference, which is a numerical probability measure on the set of deterministic preferences.

As a qualitative comparison of comparisons, the concept has many interpretations and applications, including measurement of dissimilarity, characterization of non-transitive binary relations, and others.

1. Introduction

Economic decision theory traditionally starts with a fixed preference over alternatives, and examines how that preference can be maximized.

A generalization of that approach allows preferences to be random: a preference relation is chosen at random, and then a maximizing decision is made under that chosen preference.[2] This "numerical" approach makes the assumption that the decision maker's choices are determined by the numerical probability value assigned to each deterministic preference.

In this paper, we generalize further. We study an alternative "qualitative" approach, examining the more primitive question of where the numerical probabilities might come from. We define the concept of a (relative) likelihood, $\succcurlyeq$, which is a quaternary relation over a given set of alternatives. Intuitively, we view a decision maker as considering subjective, qualitative comparison questions such as: "Is it at least as likely that alternative *a* is strictly preferred to alternative *b*, as that alternative *c* is strictly preferred to alternative *d*?" If the decision maker's answer is "yes", we will write $(a, b) \succcurlyeq (c, d)$.

We are still treating preferences as random. The phrase "as likely" connotes randomness. In our present view, randomness of whether or not a particular *a* is strictly preferred to a particular *b* stems from randomness of the underlying mechanism that selects preference relations. We take that underlying randomness as an intuitive, subjective notion.

There are many situations that our qualitative approach captures. A decision maker's preference might be state-dependent. In some states of nature (rain, sunshine, etc.), the decision maker prefers *a* to *b*, and in some other states, the decision maker prefers *b* to *a*. The decision maker may have only intuitive ideas about the likelihood of those state events, not knowing an exact numerical probability for them. Then he might only be able to say whether the chance of preferring *a* to *b* is greater than, the same as, or less than the chance of preferring *c* to *d*. This yields what we will call a likelihood relation $\succcurlyeq$. We may call it a "qualitative", "intuitive", "subjective", or "non-numerical" relation, to distinguish it from such numerical notions as traditional numerical probability measures.

The main purpose of this paper is to investigate the conditions under which a subjective, non-numerical likelihood relation $\succcurlyeq$ can be expressed numerically, i.e., can be represented by a probability measure π on the set of preferences. As a representation of $\succcurlyeq$, the probability measure π assigns to each preference relation r a probability $\pi(r)$ in a way that quantifies the qualitative relation $\succcurlyeq$. For example, suppose that $(a, b) \succcurlyeq (c, d)$, i.e., it is at least as likely that alternative *a* is strictly preferred to alternative *b*, as that alternative *c* is strictly preferred to alternative *d*. Then, the set of

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1 Marcel K. Richter passed away on July 11, 2014.

2 Cf. Block and Marschak (1960), Barberá and Pattanaik (1986), McFadden and Richter (1971, 1990), McFadden (2005), and Bandyopadhyay et al. (1999, 2004).

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preference relations r ranking a higher than b has a π-probability that is as great as that of the set of preference relations r ranking c higher than d.

We could then view the subjective likelihood $\succcurlyeq$ as a precursor to the numerical probability measure π. And we could apply numerical techniques to solve decision making problems presented by intuitive likelihoods. Our goal is to find conditions when this is possible. The analogue with numerical utility theory as a proxy for intuitive preference relations should be clear.

Simple examples will show that not every likelihood is representable by a probability measure. In our Theorem 1, we provide natural axioms on likelihoods $\succcurlyeq$, and prove that they characterize those qualitative likelihoods that can be represented by a positive numerical probability measure. We do the same for not-necessarily-positive representations in Theorem 2.

As a qualitative comparison of comparisons, the concept has many applications, including measurement of dissimilarity, and characterization of non-transitive binary relations. We discuss these applications in Section 3.

Our work relates to contributions by several other authors.

Problems of representing an intuitive probable relation have been studied by Kraft et al. (1959), Scott (1964), Chateauneuf (1985), and others. While they considered a binary relation over a Boolean algebra of subsets of a given set, we consider a quaternary relation over the given set. They sought a probability measure on the given set, whereas we seek a probability measure on the set of preferences over the given set.

In their work on representing a quaternary relation, Scott (1964), Debreu (1960), and Fuhrken and Richter (1991) sought an additive utility function. Our work considers a likelihood, which is also a quaternary relation, but we seek representation by stochastic preference rather than by additive utility function.

Problems of revealed stochastic preference have been studied by Block and Marschak (1960), McFadden and Richter (1971, 1990), Barberá and Pattanaik (1986), McFadden (2005), and others. They started with a numerical probabilistic choice function, which assigns probabilities of being chosen to alternatives in given opportunity sets, and then sought a stochastic preference to rationalize the choice function. We also seek a stochastic preference, but we want it to explain a qualitative likelihood relation rather than a numerical probabilistic choice function.

Bervoets and Gravel (2007) used a quaternary relation to model dissimilarity. Our likelihood is also a quaternary relation. As Section 3 shows, our work provides a new measurement of dissimilarity.

2. Representability

Throughout this paper, we consider an arbitrary finite set X as a universe of "alternatives", among which the decision maker might choose.

We will work with two classes of relations. The first includes binary relations r over the universe X, with the usual ordinal preference interpretation for a decision maker.

The second class includes quaternary relations $\succcurlyeq$ over the universe X. Each will express a decision maker's qualitative beliefs about preferences—not whether alternative a is preferred to alternative b, but whether that is intuitively more likely than c being preferred to d.

Rather than treating $\succcurlyeq$ as a quaternary relation over the universe X, we usually treat it as a binary relation over $X \times X$. Then, we will study binary "likelihood" comparisons between pairs like (a, b) and (c, d). Our interpretation of pairs like (a, b) will be in terms of preference: (a, b) signifies that "a is strictly preferred to b".[3] We interpret "$(a, b) \succcurlyeq (c, d)$" as "yes" to the question:

"Is it at least as likely that alternative a
is strictly preferred to alternative b,
as that alternative c is strictly preferred
to alternative d?" (1)

2.1. Relations

We begin with the first relation type, ordinal preferences.[4]

A *weak preference* is a binary relation r on X that is reflexive, transitive, and total. We denote by $\mathcal{R}$ the set of weak preferences over X. For brevity, we often drop the term "weak". For any preference r, we denote its strict (asymmetric) part by P_r, and its indifference (symmetric) part by I_r.[5]

What do we mean when we say that a preference for a over b is random? In our view, the essential randomness lies in the selection of the preferences r, as expressed in the next definition, where we describe the randomness by a probability measure on the Boolean algebra $\mathcal{A}$ of all subsets of $\mathcal{R}$.

A *stochastic preference* is a probability measure π on $(\mathcal{R}, \mathcal{A})$. We say π is *positive* if $\pi > 0$, i.e., $\pi(r) > 0$ for all $r \in \mathcal{R}$.

Thus, we are viewing the binary rankings $r \in \mathcal{R}$ as states of nature, and all the subsets of $\mathcal{R}$ as events.

After ordinal preference rankings r and their stochastic counterparts π are discussed, we introduce our second type of relation, which is a "higher level" one, of comparing comparisons: it compares binary preference comparisons like (a, b) and (c, d).

Definition 1 (*Likelihood*). A *likelihood* $\succcurlyeq$ is a binary relation on $X \times X$ that is reflexive and transitive.

As noted earlier, we interpret $(a, b) \succcurlyeq (c, d)$ as "yes" to question (1). We do not require that the decision maker always be able to answer these questions, as a likelihood $\succcurlyeq$ is not necessarily a total relation.

We denote the strict (asymmetric) part of a likelihood $\succcurlyeq$ by $\succ$. Then $\succ$ is transitive and asymmetric. We also denote the indifference (symmetric) part of $\succcurlyeq$ by $\sim$. Then $\sim$ is transitive, symmetric, and reflexive.[6]

Next we will discuss probability representations of the notion $(a, b) \succcurlyeq (c, d)$.

2.2. Representations

Main question Can we represent intuitive likelihood comparisons by numerical probability comparisons?

In our view, such a representation is a probability measure on the set $\mathcal{R}$ that can replace the intuitive "at least as likely" in (1) by

[3] Indeed, there are several alternative interpretations of (a, b): (i) "a is strictly preferred to b", (ii) "a is weakly preferred to b", (iii) "utility of a is higher than that of b", and (iv) "utility of a is as high as that of b". The present paper uses (i). Indeed, approaches using (i) and (ii) are dual to each other. Approaches using (i) and (iii) are equivalent. Approaches using (ii) and (iv) are also equivalent. In Richter and Wong (2013), we considered all of these variants and discussed their relations.

[4] For any set Y (e.g. $Y = X$, or $X \times X$), a binary relation r over Y is *reflexive* if $a\,r\,a$ for all $a \in Y$; it is *transitive* if for all $a, b, c \in Y$, $[(a\,r\,b)\&(b\,r\,c)] \Rightarrow (a\,r\,c)$; it is *total* if for all $a, b \in X$ with $a \neq b$, either $a\,r\,b$ or $b\,r\,a$; it is *complete* if it is reflexive and total; it is *symmetric* if for all $a, b \in Y$, $(a\,r\,b) \Rightarrow (b\,r\,a)$; and it is *asymmetric* if for all $a, b \in Y$, $(a\,r\,b) \Rightarrow \neg(b\,r\,a)$.

[5] To emphasize the "weak" feature of r, we may write R_r instead of r (writing "aR_rb", instead of "$a\,r\,b$"). Thus, for all $a, b \in X$, we have: aP_rb if and only if $[(a\,R_r\,b)\ \&\ \neg(b\,R_r\,a)]$, and aI_rb if and only if $[(a\,R_r\,b)\&(b\,R_r\,a)]$.

[6] Moreover, $\succ$ and $\sim$ are compatible in the sense that: for all $a, b, c, d, e, f \in X$, if $(a, b) \sim (c, d) \succ (e, f)$, then $(a, b) \succ (e, f)$, and if $(a, b) \succ (c, d) \sim (e, f)$, then $(a, b) \succ (e, f)$.