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Stochastic stability in assignment problems☆

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ABSTRACT

In a dynamic model of assignment problems, it is shown that small deviations suffice to move between stable outcomes. This result is used to obtain no-selection and almost-no-selection results under the stochastic stability concept for uniform and payoff-dependent errors. There is no-selection of partner or payoff under uniform errors, nor for agents with multiple optimal partners under payoff-dependent errors. There can be selection of payoff for agents with a unique optimal partner under payoff-dependent errors. However, when every agent has a unique optimal partner, almost-no-selection is obtained.

1. Introduction

We study two-sided one-to-one matching markets with side payments. Two-sided matching markets with side payments – *assignment problems* – were first analyzed by Shapley and Shubik (1971). In an assignment problem, indivisible objects (e.g., jobs) are exchanged with monetary transfers (e.g., salaries) between two finite sets of agents (e.g., workers and firms). Agents are heterogeneous in the sense that each object may have different values to different agents. Each agent either demands or supplies exactly one unit. Thus, agents form pairs to exchange the corresponding objects and at the same time make monetary transfers.

An outcome for an assignment problem specifies a matching between agents of both sides of the market and, for each agent, a payoff. An outcome is in the *core* if no coalition of agents can improve their payoffs by rematching among themselves.

This paper adds to the literature on the dynamics of assignment problems. It has been recently shown that under plausible dynamics of rematching and surplus sharing, convergence to the core of the assignment problem is assured (Chen et al., 2012; Biró et al., 2013; Klaus and Payot, forthcoming; Nax et al., 2013). A typical such dynamic involves two agents meeting every period, and if they can improve upon their current payoffs by matching with one another, they do so.[1] The current paper analyzes the effect of perturbations which can move the process away from core outcomes. Under such perturbations, any agent can occasionally make an error and move to an outcome which gives him a payoff lower than his current payoff. Take any core outcome and subject it to a small deviation within a single matched pair whereby one of the agents in the pair gains a unit of payoff and the other loses a unit of payoff. It is shown that such a small deviation suffices for the unperturbed blocking dynamics to subsequently move to another core outcome. More specifically, this can occur in a way that the reached optimal matching is the same as the original matching and only payoffs change (Theorem 2), or in such a way that payoffs stay the same and a different optimal matching, if one exists, is reached (Theorem 3).

In much the same way that perfectness concepts can be used to analyze the robustness of static equilibria to mistakes (Selten, 1975; Myerson, 1978), when perturbations are added to a dynamic process, more precise predictions about the long term behavior of the process can be obtained. Specifically, the invariant measure

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[1] There has long existed a literature on paths to stability in matching problems with non-transferable utility. See, for example, Roth and Vande Vate (1990), Diamantoudi et al. (2004), Klaus and Klijn (2007) and Kojima and Ünver (2008). It is therefore notable that similar problems have only recently begun to be addressed for matching problems with transferable utility.

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of the perturbed process can place much greater weight on some equilibria of the unperturbed process than on others. As the probability of perturbations is taken to zero, the equilibria that have positive weight under the limiting invariant measure are known as *stochastically stable* equilibria. These are the outcomes that we would expect to observe most frequently in the long run when perturbations are rare. An example of such a process is the best-response dynamic (the dynamic justification for Nash equilibrium) together with some small probability that any given agent makes a mistake and does not play a best response. Recent experimental evidence supports perturbed best response as a behavioral rule, but is mixed as to whether mistakes depend strongly on associated payoff losses. Mäs and Nax (2014) find evidence that mistakes that cause greater payoff loss are less likely, whereas Lim and Neary (2013) find this not to be the case. The results of the current paper cover both of these possibilities.

A consequence of small deviations sufficing to move the process between core states is that stochastic stability is a weak selection concept. Results are first derived for two simple error processes, *uniform* and *stepped*, which occur in the literature. Following this, we define the class of *weakly payoff monotone* error processes. This is a large class of processes, under which errors that lead to greater payoff losses are (weakly) less likely. It is shown that for every process in this class, the results for either uniform or stepped errors pertain.

Uniform errors (see Young, 1993) are such that every error has the same (order of magnitude of) probability of occurring. Under uniform errors we show that there is no selection: every core state is stochastically stable (Theorem 5). This result is similar in spirit to the results of Jackson and Watts (2002) and Klaus et al. (2010) which find no selection in marriage and roommate problems under uniform error processes.[2]

Payoff-dependent errors occur with probabilities that depend on the payoff loss incurred when they are made.[3] Logit errors (see Blume, 1993) are an example of such errors, and occur with probabilities that are log-linear in such payoff losses. Another possibility is that errors involving indifference occur more often than errors by which agents' payoffs strictly decrease (Serrano and Volij, 2008). We refer to these latter errors as *stepped*. Under stepped errors we find that

(i) All optimal matchings occur in some stochastically stable outcome (Theorem 6).
(ii) For any agent with different partners in different optimal matchings, any core payoff can be attained as a stochastically stable payoff (also Theorem 6).
(iii) For agents who have the same partner in every optimal matching, the set of stochastically stable payoffs can be a strict subset of the set of core payoffs (Example 3), but
(iv) if every agent has a unique optimal partner, then we obtain almost-no-selection: the interior of the set of core payoffs is stochastically stable, where the interior refers to the set of payoffs for which no two agents who are not matched to one another at the optimal matching can do at least as well by matching with one another (Theorem 7).

Finally, we define the class of *weakly payoff monotone* error processes. This class is very large. Despite this, every error process in this class is either similar to uniform errors or to stepped errors (Theorem 8). If errors involving small payoff losses occur just as often as errors involving indifference, the result for uniform errors, Theorem 5, holds. If errors involving small payoff losses occur less often than errors involving indifference, then the results for stepped errors, Theorems 6 and 7, hold. Importantly, this latter class includes adaptations of popular choice rules, logit and probit choice, that are derived from random utility models.

2. Related literature

2.1. Perturbed dynamics and selection in the core

A related literature is the literature on *convergence to the core* in cooperative games (Feldman, 1974; Green, 1974; Sengupta and Sengupta, 1996; Agastya, 1997; Serrano and Volij, 2008; Newton, 2012b). Agastya (1999) shows that if a cooperative game is modeled as a generalized Nash demand game, then the stochastically stable states are states in the core at which the maximum payoff over all agents is minimized. Newton (2012b) shows that, under some conditions, the addition of joint strategic switching to such models leads to Rawlsian selection within the strong core (referred to as the 'interior core' in the cited paper), maximizing the minimum payoff over all agents. In assignment problems, the strong core is empty as value function inequalities for matched pairs always hold with equality, so the methods of Newton (2012b) cannot be applied. Nax and Pradelski (2015) have recently shown a maxmin selection result within the core for assignment games, a result discussed next.

2.2. Nax and Pradelski (2015)

Nax and Pradelski (2015) analyze an error process in which payoffs can be shocked with a probability which is log-linear in the size of the shock. If an agent, following a shock, has a payoff lower than that which he could achieve by a change of partner, then he can change his partner. Using arguments adapted from Newton and Sawa (2015), Nax and Pradelski (2015) show that under this process the set of stochastically stable states is a subset of the least core (Maschler et al., 1979). If any agent has multiple optimal partners, then the least core equals the core, so the error processes in the current paper also select within the least core. If every agent has a unique optimal partner, then the least core can be a strict subset of the interior of the core, therefore in this case the least core inclusion of Nax and Pradelski (2015) fails under the error processes of the current paper. The reason for the difference between the two papers is that the current paper allows for errors whereby the agents in a given pair remain matched yet adjust the payoffs they obtain within the pair, whereas the cited paper only considers errors with sufficient strength to cause a pair to break up. This restriction brings their model closer to models in the NTU literature, which is discussed next.

2.3. Selection in matching problems

Newton and Sawa (2015) give a general selection result for matching problems (marriage problems, roommate problems, college admissions problems) for any error process, including payoff-dependent processes. All stochastically stable matchings lie within the set of matchings which are most robust to one-shot deviation. This is often a strict subset of the core. It is worth commenting on why selection is not often likewise attained for assignment problems under payoff-dependent dynamics. The reason is that in the assignment problem, there is always another core outcome in which payoffs do not differ at all, or differ only slightly, from the payoffs of the current outcome. If any agent

[2] The marriage problem is the non-transferable utility equivalent of the assignment problem.

[3] The relation of such rules to uniform mistake models can be thought of as similar to the relation between the static concepts of Proper Equilibrium (Myerson, 1978) and Trembling Hand Perfect Equilibrium (Selten, 1975). In the former, mistakes associated with larger payoff losses are less likely, whereas in the latter there is no difference.