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Risk neutrality regions

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ABSTRACT

An Expected Utility maximizer can be risk neutral over a set of nondegenerate multivariate distributions even though her NM (von Neumann Morgenstern) index is not linear. We provide necessary and sufficient conditions for an individual with a concave NM utility to exhibit risk neutral behavior and characterize the regions of the choice space over which risk neutrality is exhibited. The least concave decomposition of the NM index introduced by Debreu (1976) plays an important role in our analysis as do the notions of minimum concavity points and minimum concavity directions. For the special case where one choice variable is certain, the analysis of risk neutrality requires modification of the Debreu decomposition. The existence of risk neutrality regions is shown to have important implications for the classic consumption–savings and representative agent equilibrium asset pricing models.

1. Introduction

Standard textbook treatments of the economics of risk typically show that an Expected Utility maximizer will be risk neutral in the univariate case if and only if her NM (Neumann and Morgenstern, 1953) index is linear (e.g., Mas-Colell et al., 1995, p.186). However for bivariate lotteries or distributions, the assumption that the NM utility takes the following linear form

$$U(c_1, c_2) = \alpha c_1 + \beta c_2 + \gamma, \tag{1}$$

where c_1 and c_2 denote units of two different commodities, is only sufficient and not necessary for risk neutrality.[1] An individual with a concave NM index not taking the form (1) can be risk neutral toward subsets of multivariate lotteries. We refer to these subsets as risk neutrality regions of the full choice space of lotteries or distributions.[2]

In this paper, individuals are assumed to possess Expected Utility preferences. We derive necessary and sufficient conditions for when a consumer with an NM index not taking the linear form (1) is risk neutral toward a subset of lotteries and characterize the regions of the choice space in which risk neutrality is exhibited. Since the results for the bivariate case naturally generalize to multivariate distributions, we focus on just the simpler bivariate case.[3] We also consider the important special case where one attribute is certain and the second is random.

The existence of risk neutrality regions can have important implications for popular application problems where the NM index is concave and does not take the form (1). First, suppose a consumer faces the classic two period consumption–savings, capital risk problem with a single risky asset. In response to a pure increase

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[1] Whereas the discussion of risk neutrality is commonplace for the case of univariate distributions, the multivariate case is much less thoroughly investigated. One interesting exception is Safra and Segal (1993), who consider multivariate risk neutrality for non-Expected Utility preferences.

[2] In a number of papers that have sought to extend the notion of risk aversion to multivariate Expected Utility preferences, the authors have tended to define risk attitudes in terms of utility indices. Reference to risk neutrality arises as the extreme of an individual being both risk averse and risk prone. See, for instance, Duncan (1977), Karni (1979), Kihlstrom and Mirman (1974) and Hellwig (2004).

[3] In the online Appendix L, we consider a specific example which illustrates how several of the key concepts investigated in this paper extend to more than two choice variables.

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in the risk of the asset's return (i.e., a mean preserving spread), optimal savings can remain unchanged which is consistent with risk neutral behavior. Second assume a two period representative agent exchange economy in which there exists one risky and one risk free asset. Then for a particular endowment of period one consumption, the equilibrium risk premium can go to zero which is consistent with the representative agent being risk neutral.

A key element in our analysis is Debreu's (1976) classic decomposition of a concave, multicommodity NM index into a least concave utility representing certainty preferences over commodity vectors and a univariate concave transformation reflecting risk preferences. Debreu's focus was on proving the existence of a least concave utility given that a concave NM utility is known to exist. However in order to distinguish the specific set of lotteries toward which the consumer is risk neutral, one must go beyond existence and actually derive from a given concave U the specific form of the least concave utility. One must also identify the set of minimum concavity points and minimum concavity directions (characterizing where and in which directions the Hessian of the least concave utility vanishes). The very popular homothetic and quasihomothetic NM utilities[4] permit particularly clear characterizations of the subset of risk neutral lotteries. This follows from the very special properties of the minimum concavity points and minimum concavity directions of these utilities (see Kannai and Selden, 2014). However we emphasize that risk neutrality regions also occur for non-homothetic and non-quasihomothetic preferences.

To extend our analysis of risk neutrality to the case where one of the commodities is certain, it is necessary to modify Debreu's decomposition result. For a given NM index, the set of minimum concavity points, minimum concavity directions and least concave utilities can differ when one commodity is certain versus when no commodity is certain. As a result, the risk neutrality regions will typically change. As we discuss in Section 5, the fact that the least concave utilities can diverge when one of the commodities become certain seems to have been missed in the literature. Not recognizing this point can result in decompositions of a given NM index into certainty preferences and risk preferences which are erroneous and ultimately lead to incorrect behavioral predictions such as how an individual will react to increasing risk.

The rest of the paper is organized as follows. In the next section, we give two motivating examples. The first illustrates the existence of risk neutrality regions in a standard lottery choice setting. The second demonstrates that risk neutrality regions can have interesting implications for optimal savings behavior. Section 3 reviews the definitions of risk neutrality toward univariate and bivariate probability distributions. In Section 4, we first discuss the notions of least concave utility, minimum concavity points and minimum concavity directions and then use them to characterize the subsets of distributions where an individual will be risk neutral. Section 5 considers the special case where one preference attribute is certain. Section 6 provides two additional economic applications. Selected proofs are provided in the Appendices A–E of this paper and the remaining proofs and supplemental materials are available online (http://dx.doi.org/10.1016/j.jmateco.2015.10.010).

2. Motivating examples

Before formally defining risk neutrality, we consider two motivating examples which illustrate that an individual or an economy can exhibit risk neutral behavior even though the assumed bivariate NM index does not take the linear form of Eq. (1).[5]

The following example shows that an individual can be risk neutral toward some (but not all) lotteries if her NM index does not take the linear form (1).

Example 1. Assume the individual's NM index is given by

$$U(c_1, c_2) = 600c_1 + 600c_2 - (c_2 - c_1)^2 - (c_2 - c_1)^4, \tag{2}$$

where $(c_1, c_2) \in (0, 5)^2$. It can be verified that this utility function is strictly increasing and concave, implying that the indifference curves are well behaved (convex) in $(0, 5)^2$. Consider the lottery $L_1 = \left((1, 1.5), \frac{1}{2}; (2, 2.5), \frac{1}{2}\right)$, where $(1, 1.5)$ and $(2, 2.5)$ are the vector payoffs and $\frac{1}{2}$ is the probability of each payoff. $L_2 = ((1.5, 2), 1)$ is a degenerate lottery with its payoff equal to the means of the payoffs of L_1. Following Safra and Segal (1993), an individual is said to be risk neutral toward the risky lottery L_1 if she is indifferent between it and the special degenerate lottery L_2 (see Definition 2 below). Lotteries L_1 and L_2 are plotted in Fig. 1 and lie on the common ray

$$c_2 = c_1 + 0.5.$$

Using the NM index (2), computation of the Expected Utility for L_1 and L_2 yields

$$\begin{aligned} EU(L_1) &= \frac{1}{2}(600 + 900 - 0.25 - 0.0625) \\ &\quad + \frac{1}{2}(1200 + 1500 - 0.25 - 0.0625) = 2099.6875 \end{aligned}$$

and

$$EU(L_2) = 900 + 1200 - 0.25 - 0.0625 = 2099.6875.$$

Since $EU(L_1) = EU(L_2)$, the individual is risk neutral toward L_1. We next show that although the individual is risk neutral toward L_1, she is not risk neutral toward all lotteries since the NM index (2) does not take the linear form (1). Consider lotteries $L_3 = \left((1, 1.5), \frac{1}{2}; (2, 3.5), \frac{1}{2}\right)$ and $L_4 = ((1.5, 2.5), 1)$, where L_4 is the degenerate lottery with payoff corresponding to the means of the payoffs of L_3. Lotteries L_3 and L_4 also lie along a common ray in $c_1 - c_2$ payoff space which is defined by

$$c_2 = 2c_1 - 0.5.$$

However in this case since

$$\begin{aligned} EU(L_3) &= \frac{1}{2}(600 + 900 - 0.25 - 0.0625) \\ &\quad + \frac{1}{2}(1200 + 2100 - 2.25 - 5.0625) = 2396.1875 \end{aligned}$$

and

$$EU(L_4) = 900 + 1500 - 1 - 1 = 2398,$$

the individual is not risk neutral but rather is risk averse toward L_3 since her Expected Utility is lower for L_3 than for L_4.

The next example is based on the classic two period consumption–savings, capital risk problem. Certain first period and random second period consumption are denoted, respectively, by c_1 and $\widetilde{c}_2$. In period one, the consumer is endowed with income I and chooses how much to consume c_1 and save $I - c_1$. Saving takes place via a

[4] The terms homothetic and quasihomothetic are defined as customary (see Deaton and Muellbauer, 1980, pp. 143–145). It should be noted that in the Expected Utility setting, if the NM index is a member of the HARA (hyperbolic absolute risk aversion) family of utilities, then preferences are homothetic or quasihomothetic.

[5] As noted in footnote 10, the case where the NM index is linear over a portion of its domain and the payoffs corresponding to a distribution or lottery are defined on this subdomain will not be distinguished from the case where the index is linear over its entire domain.