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Decomposing random mechanisms☆

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ABSTRACT

Random mechanisms have been used in real-life situations for reasons such as fairness. Voting and matching are two examples of such situations. We investigate whether the desirable properties of a random mechanism survive decomposition of the mechanism as a lottery over deterministic mechanisms that also hold such properties. To this end, we represent properties of mechanisms – such as ordinal strategy-proofness or individual rationality – using linear constraints. Using the theory of totally unimodular matrices from combinatorial integer programming, we show that total unimodularity is a sufficient condition for the decomposability of linear constraints on random mechanisms. As two illustrative examples we show that individual rationality is totally unimodular in general, and that strategy-proofness is totally unimodular in some individual choice models. We also introduce a second, more constructive approach to decomposition problems, and prove that feasibility, strategy-proofness, and unanimity, with and without anonymity, are decomposable in non-dictatorial single-peaked voting domains. Just importantly, we establish that strategy-proofness is not decomposable in some natural problems.

1. Introduction

Random mechanisms are frequently used in sustaining fairness among market participants. For example, admission to public schools through school choice in the US (cf. Abdulkadiroğlu and Sönmez, 2003) is administered in many districts through centralized random mechanisms that use random tie-breakers. Some voting and social-choice environments also use random mechanisms. Jury selection, draft lotteries, and ballot positioning are further examples (cf. Fishburn, 1984). Other examples include voting in Olympic figure skating competitions and the election method of military leaders (known as doges) in Venice (used for more than 500 years; cf. Lines, 1986). Some random mechanisms are designed directly to use a lottery over predetermined deterministic mechanisms, as in school choice. Another approach in random mechanism design uses probabilistic assignment over outcomes for each situation rather than deterministic mechanisms in the support of the random mechanism. Competitive equilibrium from equal incomes of Hylland and Zeckhauser (1979), the probabilistic serial mechanism of Bogomolnaia and Moulin (2001) for object allocation, and maximal lottery methods (cf. Kreweras, 1965; Fishburn, 1984) for voting are some examples of this approach.

Random mechanisms correspond to the full range of possible mechanisms. From the point of view of mechanism design, they cannot be neglected in the search for the best mechanism to implement a desired goal. On the other hand, many market design situations require transparency of the mechanism. Randomness of a mechanism is often a source of additional complexity in explaining and educating the agents who will participate in its implementation. Although simple tie-breakers can easily be explained to the participants in certain situations (e.g., in school choice), more complex random mechanism implementation often hinges on the condition that we can implement a deterministic mechanism to represent the random mechanism. For this reason, the market designer may want to resolve the uncertainty regarding the mechanism as soon as possible, before the participants' private information is collected. Thus, the representability of a random mechanism as a randomization over deterministic mechanisms that also have the same properties could be crucial to the success of the design.

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When a property is transferable through decomposition, it holds both ex ante, i.e., before the uncertainty regarding the mechanism is resolved, and ex post, i.e., after this uncertainty is resolved. In this case, the mechanism is more robust and is not affected by the market participants' access to information regarding the resolution of the uncertainty in the mechanism. For example, if dominant-strategy incentive-compatibility (or strategy-proofness) is decomposable, then it is best for an agent to reveal his preferences truthfully regardless of if all he knows is that a stochastically strategy-proof mechanism will be implemented or if he knows exactly, after the lottery is resolved, which strategy-proof deterministic mechanism will be implemented. If such a decomposition goes through, this deterministic mechanism, in many cases, can be explained more transparently to the participants.[1]

The goal of this paper is to narrow the gap between our understanding of random and deterministic mechanisms in ordinal environments. Although we have a good understanding of which properties of deterministic mechanisms are preserved when we randomize over deterministic mechanisms, the other direction remains quite unclear. Exploring the possibility (or impossibility) of decomposition of a property will show whether, without loss of generality, we can focus on lotteries over deterministic mechanisms in mechanism design.

We adopt two approaches in determining the decomposability of properties of random mechanisms. We start with formulating a simple sufficient condition and then use a constructive approach for more complex properties where this first approach is inconclusive.

First, we reformulate a useful approach to mechanism design that has been used in combinatorial integer programming in various applications. We show how to analyze which properties of a random mechanism are decomposable by employing *totally unimodular* (*TUM*) *decomposition* (cf. Theorem 1). In this way, we contribute to the growing literature on new approaches to mechanism design using linear programming tools, which have recently found their way into mainstream economics (see Vohra, 2011). Using these methods, we show that every individually rational random mechanism is a lottery over individually rational deterministic mechanisms in a variety of environments including object allocation, social choice, and matching (cf. Theorem 2). Strategy-proofness with and without individual rationality constraints are also TUM in certain models. We give an example of an individual choice model where strategy-proofness is TUM and hence decomposable (cf. Theorem 3).[2,3]

Surprisingly, we find a counterexample in which even with a single agent, in the universal house allocation or voting domains, strategy-proofness is not decomposable and hence not TUM (cf. Theorem 4). On the other hand, together with other properties, strategy-proofness can still be decomposable in these domains.

Moreover, TUM decomposability is sometimes too strong. Even though a property is not TUM, it could still be decomposable. For example, it is straightforward to show that in the single-peaked voting domain (and hence in the universal domain), strategy-proofness, unanimity, and feasibility taken together are not TUM.[4,5] Despite this fact, we prove that they are decomposable (cf. Theorem 5). In proving this result, we employ a constructive approach, which requires the knowledge of the characterization of deterministic mechanisms that carry the same properties as the random mechanism. Using this information, we construct a lottery over deterministic mechanisms with the required properties that induces a given random mechanism. Moreover, we prove that strategy-proofness is decomposable for tops-only mechanisms (i.e., when the mechanism outcome relies only on the reported top choices of the agents) in a single-peaked voting domain and unanimity is not needed for this result as an additional property (cf. Theorem 6).

As a corollary to the proof of the decomposability of strategy-proofness and unanimity in a single-peaked voting domain, we also establish that anonymity, unanimity, strategy-proofness, and feasibility are jointly decomposable (cf. Theorem 7).[6]

A forerunner to our work, Gibbard (1977) studied the decomposition of strategy-proofness in voting when all strict preference rankings are admissible, i.e., in the universal social-choice domain. In this model, he showed that any unanimous and strategy-proof random mechanism is a randomization over unanimous and strategy-proof deterministic mechanisms. Such deterministic mechanisms are known to be dictatorships (cf. Gibbard, 1973; Satterthwaite, 1975). The question of whether such a decomposition is possible in restricted domains in which there are non-dictatorial unanimous and strategy-proof deterministic mechanisms has remained open. Using our tools, we answer it in the affirmative in the single-peaked voting domain. Deterministic strategy-proof and unanimous mechanisms in this domain were characterized by Moulin (1980) and have been studied intensively ever since. It turns out that strategy-proofness and unanimity, with and without anonymity, are decomposable even though they are not TUM.[7] This result is surprising given the observation by Ehlers et al. (2002) that some strategy-proof and unanimous mechanisms cannot be decomposed in the same domain as a randomization over the particular subset of strategy-proof and unanimous deterministic mechanisms that they study. This paper's main contribution is the characterization of strategy-proof and unanimous random mechanisms in the single-peaked preference voting model. Unlike our approach, they come up with a random mechanism class that is not defined as a probability distribution over deterministic strategy-proof and unanimous mechanisms. Hence,

1 In algorithmic game theory, the computer science literature that deals with game theory and mechanism design, the decomposability of a property has also attracted special attention. The literature refers to the decomposability of a property as *universality* (cf. Nisan and Ronen, 1999). For example, universal strategy-proofness (or truthfulness, as sometimes referred to in the computer science literature) is inspected in a recent paper by Krysta et al. (2014) to find a matching covering as many agents as possible without sacrificing universal strategy-proofness in the house allocation problem. They find an upper approximation bound for this problem.

2 A deterministic mechanism is individually rational if its outcome is preferred by agents to their outside options. A random mechanism is individually rational if its outcome first-order stochastically dominates agents' outside options.

3 Observe that there could be other ways of proving that these properties are decomposable in the aforementioned domains. Some of these proofs are simpler and they may not need the TUM property. However, our theorems are stronger than just showing that these properties are decomposable, as we prove that they are TUM (a sufficient but not necessary condition for decomposability).

4 A deterministic mechanism is strategy-proof if, for every agent, submitting his true preference ranking is at least as good as submitting any other ranking irrespective of the preference rankings submitted by other agents. This is equivalent to ex-post incentive-compatibility. A random mechanism is strategy-proof if, for every agent, submitting his true preference ranking first-order stochastically dominates submitting any other preference ranking irrespective of the preference rankings submitted by other agents. This is the standard notion of incentive-compatibility of ordinal random mechanisms introduced by Gibbard (1977, 1978), Roth and Rothblum (1999) and Bogomolnaia and Moulin (2001).

5 Unanimity is a weak form of efficiency. A mechanism is unanimous if, whenever there are outcomes that are among the most desirable choices for all agents, then the mechanism implements one of these outcomes.

6 A mechanism is anonymous if the outcome of the mechanism depends only on the set of preferences reported, not on who reported them.

7 While proving this result, we also obtain a corollary to Moulin (1980) using our tops-onlyness results, the characterization of the full class of deterministic, unanimous, and strategy-proof voting mechanisms in a single-peaked domain (cf. Corollary 1). As far as we know, we are the first to give this full characterization.