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Potential games with incomplete preferences ☆

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ABSTRACT

This paper studies potential games allowing the possibility that players have incomplete preferences and empty best-response sets. We define four notions of potential games, ordinal, generalized ordinal, best-response, and generalized best-response potential games, and characterize them using cycle conditions. We study Nash equilibria of potential games and show that the set of Nash equilibria remains the same when every player's preferences are replaced with the smallest generalized (best-response) potential relation or a completion of it. Similar results are established about strict Nash equilibria of ordinal and best-response potential games. Lastly, we examine the relations among the four notions of potential games as well as pseudo-potential games.

1. Introduction

Whether a Nash equilibrium exists and how players can reach one are important questions in game theory. Potential games are useful in that they often guarantee the existence of a Nash equilibrium and the convergence of a dynamic adjustment process to a Nash equilibrium. Let us consider a myopic adaptive process in which players change their strategies sequentially and unilaterally to improve their payoffs. When the process stops, we obtain a Nash equilibrium since at that point no player can further improve his payoff given the play of others. Then a question that arises naturally is whether the process is guaranteed to terminate starting from an arbitrary initial point. This property is called the finite improvement property in Monderer and Shapley (1996), and they provide a class of games, namely generalized ordinal potential games, that possess this property. If the game is a generalized ordinal potential game, it is as if players jointly improve a common payoff function in the process. If in addition the game is finite, the process must stop after a finite number of steps reaching a Nash equilibrium. We can also think of a "speed-up" version of the process in which players choose their best responses whenever possible. When this new process stops, we still obtain a Nash equilibrium. Since we restrict players to deviate to best responses, a weaker notion of potential games than generalized ordinal potential games suffices to guarantee convergence to a Nash equilibrium, and this observation leads us to study the class of potential games called generalized best-response potential games.

Alternatively, we can consider a myopic adaptive process in which players take turns to deviate to a weakly better strategy. In this case, even when players reach a Nash equilibrium, the process may continue because there may be some player who has an indifferent strategy. Instead, when the process terminates, we obtain a stronger notion of Nash equilibrium, namely a strict Nash equilibrium at which no player can weakly improve his payoff given the play of others. For a generic finite ordinal potential game where there is no indifference cycle, the process converges to a strict Nash equilibrium after a finite number of steps. Similarly, we can consider a faster version of the process in which players choose their best responses whenever possible. Again, we can guarantee the convergence of this process to a strict Nash equilibrium with a weaker notion of potential games than ordinal potential games, and this motivates us to study best-response potential games.

In this paper, we study the aforementioned four classes of potential games as well as pseudo-potential games. These concepts of potential games have been developed in the existing literature. For example, ordinal and generalized ordinal potential games appear in Monderer and Shapley (1996) and Norde and Patrone (2001); best-response and generalized best-response potential games are introduced in Voorneveld (2000) and Kukushkin (2004), respectively; and pseudo-potential games are studied in Dubey et al. (2006). We consider a more general setup than those considered in the existing literature in the following two aspects.

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First, we allow that players' preferences are incomplete, while most existing work assumes complete preferences which are often represented by payoff functions. Second, we allow that players' best-response sets are empty, whereas most existing work considers the case where they are nonempty by assuming that strategy sets are finite or that strategy sets are compact and payoff functions are continuous. In many games, players have incomplete preferences or empty best-response sets. For instance, incomplete preferences arise naturally in a situation where each outcome of the game is associated with a vector of real numbers and an outcome is preferred to another if the vector associated with the former outcome dominates that with the latter in all components, as considered in Shapley (1959). Also, players in games with continuous strategy sets (for example, a Bertrand duopoly game with a homogeneous product and a Hotelling location game) often have empty best-response sets. By dropping the two assumptions commonly imposed in the existing literature, we expand the scope of games to which the potential game approach can be applied.

We first define the four classes of potential games formally (Definitions 1 and 2) taking into account the possibility of incomplete preferences and empty best-response sets. When players have incomplete preferences, it is more natural to work with preferences directly rather than with numerical representations, and thus we take a purely ordinal approach by defining potentials in terms of binary relations, as in Norde and Patrone (2001). We characterize the four classes of potential games using familiar cycle conditions (Theorem 1). The basic idea is that we can construct a potential relation if and only if the play does not cycle in the aforementioned myopic adaptive processes. Next we study Nash equilibria of potential games. Our main results (Theorem 2) can be summarized as follows.

1. If the game is a generalized best-response potential game, the set of Nash equilibria remains the same when every player's preferences are replaced with the smallest generalized best-response potential relation or a completion of it.
2. If the game is a best-response potential game, the set of strict Nash equilibria remains the same when every player's preferences are replaced with the smallest best-response potential relation, and for any strict Nash equilibrium, there exists a complete best-response potential relation such that it continues to be a strict Nash equilibrium when every player's preferences are replaced with the complete relation.

If the game is a generalized ordinal potential game, we obtain the same results about Nash equilibria as those obtained with generalized best-response potential games. This suggests that using the stronger notion of generalized ordinal potential games provides no additional information about Nash equilibria. The key idea is that, since only best responses matter for Nash equilibria, it suffices to work with the weaker concept of generalized best-response potential games that preserve players' preferences on strategy profiles that involve best responses. Similarly, ordinal potential games add no further information about strict Nash equilibria over that provided by best-response potential games. Hence, there is an advantage of using the notions of (generalized) best-response potential games over (generalized) ordinal potential games in that the former are weaker while containing all relevant information about (strict) Nash equilibria.

We also present a definition of pseudo-potential games adapted to our context. We show that the notion of pseudo-potential games is weaker than that of generalized best-response potential games and that, if the game is a pseudo-potential game, we can find a (complete) pseudo-potential relation that generates all Nash equilibria. We also characterize pseudo-potential games by showing that the game is a pseudo-potential game if and only if players' preferences can be modified so that best-response sets become weakly smaller and the modified game is a generalized best-response potential game. Generalized best-response potential games have a cleaner characterization than pseudo-potential games, and they can be used to characterize pseudo-potential games and find pseudo-potential relations.

This paper can be related to the literature on the theory of incomplete preferences. In economic theory, it is usually assumed that an agent's preferences are complete, that is, for any given two alternatives, an agent is able to express his preference for one of the two or indifference between the two. However, as argued in Aumann (1962), the completeness assumption is hard to accept from both normative and descriptive viewpoints, and indecisiveness does not necessarily conflict with rationality of a decision maker. In the literature, most existing work on incomplete preferences concerns individual decision-making problems dealing with topics such as utility representation and choice-theoretic foundations of incomplete preferences (see, for example, Ok, 2002; Eliaz and Ok, 2006 and references therein), while relatively little attention has been paid to strategic interaction among multiple agents with incomplete preferences. An exception is Bade (2005) who studies the existence and characterization of Nash equilibria with incomplete preferences (see also references in Bade, 2005, for existing work on games with incomplete preferences). Since there are challenges in representing incomplete preferences by an (expected) utility function, existing theorems to prove the existence of Nash equilibria based on fixed-point theorems become less applicable to games with incomplete preferences. Hence, when players' preferences are incomplete, potential games will play a bigger role in establishing the existence of Nash equilibria. This suggests that it is important to study potential games allowing the possibility that players are indecisive on occasion, and this paper contributes to the theory of incomplete preferences in this aspect.

The remaining part of this paper is organized as follows. In Section 2, we define the four concepts of potential games allowing incomplete preferences and empty best-response sets. In Section 3, we characterize these potential games using cycle conditions. In Section 4, we examine Nash equilibria of potential games. In Section 5, we investigate the relations among various classes of potential games.

2. Potential games with incomplete preferences

A strategic game consists of three components $\langle N, (X_i), (\succsim_i)\rangle$. N is a finite set of players. For each player $i \in N$, X_i is a nonempty set of strategies available to player i, and $\succsim_i$ is player i's preference relation defined on the set of strategy profiles $X = \prod_{j\in N} X_j$. For any player i, we define $X_{-i} = \prod_{j\in N\setminus\{i\}} X_j$ and write a strategy profile $x = (x_j)_{j\in N} \in X$ as (x_i, x_{-i}) where $x_i \in X_i$ and $x_{-i} \in X_{-i}$. A usual assumption on players' preferences is that each $\succsim_i$ is a complete, reflexive, and transitive binary relation on X. Throughout this paper, we drop the completeness assumption and assume that each $\succsim_i$ is a reflexive and transitive binary relation (i.e., a preorder) on X. In other words, we allow the possibility that a player is unable or unwilling to compare some strategy profiles.

Given a binary relation $\succsim$, we denote its asymmetric and symmetric parts by $\succ$ and $\sim$, respectively. That is, define $x \succ y$ if and only if $x \succsim y$ and not $y \succsim x$; and define $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$. The asymmetric part of a preference relation is called the strict preference relation, and the symmetric part is called the indifference relation. Since $\succsim_i$ is a preorder, $\succ_i$ is irreflexive and transitive (i.e., a strict order) while $\sim_i$ is reflexive, symmetric, and transitive (i.e., an equivalence relation).

We introduce several concepts of potential games for strategic games with incomplete preferences. In order to avoid the issue of numerical representation by a function and to provide a more