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Scoring rules over subsets of alternatives: Consistency and paradoxes

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ABSTRACT

We know since the works of Gehrlein and Fishburn (1980, 1981), Fishburn (1981) and Saari (1987, 1988, 1990) that, the collective rankings of scoring rules are not stable when some alternatives are dropped from the set of alternatives. However, in the literature, attention has been mainly devoted to the relationship between pairwise majority vote and scoring rules rankings. In this paper, we focus on the relationships between four-candidate and three-candidate rankings. More precisely, given a collective ranking over a set of four candidates, we determine under the impartial culture condition, the probability of each of the six possible rankings to occur when one candidate is dropped. As a consequence, we derive from our computations, the likelihood of two paradoxes of committee elections, the Leaving Member Paradox (Staring, 1986) and the Prior Successor Paradox which occur when an elected candidate steps down from a two-member committee.

1. Introduction

One of the main objectives of the social choice theory is the aggregation of individual preferences into a collective ranking that is the determination of a complete order over the set of the alternatives or candidates. This objective can be achieved when every voter gives points to each candidate in accordance with his preference order. Hence, the candidate with the highest total score will be ranked at the top of the collective ranking and the one with the lowest score will be at the bottom. This procedure defines the class of scoring rules. The Plurality rule, the Antiplurality rule and the Borda rule are, among others, some well known scoring rules that can be used for such an objective. With the Plurality rule, a candidate's score is the number of times he is top ranked in the individual rankings. With the Antiplurality rule, a candidate's score is equal to the number of voters that do not rank him last in their rankings. With m candidates, the Borda rule awards $m - j$ points to a candidate each time, he is ranked jth in a voter's ranking. The total number of points received by a candidate defines his Borda's score.

In this context, some issues have been addressed regarding the stability of a collective ranking over subsets of candidates. How a collective ranking can be altered after one or more candidates are removed from the competition? Is the new collective ranking consistent with the former one?

These issues become particularly important in the context of committee elections. Assume that, when electing a committee of size g, this committee is made by the candidates with the g greatest scores (the g top ranked candidates of the collective ranking). Staring (1986) pointed out that, when electing committees, if a member of an elected committee leaves, a new ballot, *ceteris paribus*[1], could lead to a new committee without some members of the previous ones; even worse, the two committees may be disjoint. This is called the *Leaving Member Paradox* (LMP)[2]. In this paper, we define another paradox of committee elections which is less severe than the LMP: the *Prior Successor Paradox* (PSP). Since the elected committee is formed by the candidates with the g greatest scores, we define the Prior Successor as the candidate with the $g + 1$th best score. The PSP occurs if after a member of the elected committee leaves, a new ballot leads *ceteris paribus* to a committee containing all the $g - 1$ members of the previous committee without the Prior Successor. The LMP and the PSP, as

[1] This means that voters keep their preferences unchanged on the rest of candidates no matter who is the leaving candidate.

[2] An example illustrating the LMP is provided by Staring (1986) for a voting system under which voters vote for as many candidates as there are seats to be filled.

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defined, are two inconsistencies that may tarnish the image of some voting systems. One of the objective of this paper is to study the occurrence of these paradoxes.

The origin of the questions on the stability of a collective ranking over subsets of candidates can be tracked back to the Borda–Condorcet debate. At the end of the 18th century, Borda (1781) and Condorcet (1785), who were members of the Paris Royal Academy of Sciences, proposed alternative voting rules to the one that was in use in the academy (see McLean and Urken, 1995). The *Borda rule* picks as the winner, the candidate with the highest *Borda's score*. Condorcet (1785) criticized the *Borda* rule in that it can exist a candidate that is preferred by more than half of the electorate to the *Borda winner*. Condorcet (1785) proposed a rule based on pairwise comparisons.[3] According to this rule, a candidate should be declared the winner if he beats all the other candidates in pairwise majority; such a candidate is called the *Condorcet winner*. Nonetheless, the *Condorcet principle* has a main drawback: it can lead to *majority cycles*[4] in some circumstances. Though the Borda–Condorcet debate[5] about the choice of the best voting rule is still alive, everyone agrees that they were the first authors who emphasized the fact that *scoring rules* and *pairwise comparisons* do not always lead to the same ranking.

The Borda–Condorcet debate can be envisioned in a broader picture: what are the relationships between the rankings on a set of candidates A and the rankings on the subset B included in A for a given preference profile when we use a scoring rule? Apart from classical studies analyzing the relationships between pairwise voting and scoring rules (see Dodgson, 1876; Nanson, 1882; Smith, 1973; Fishburn and Gehrlein, 1976), the first extension of these contributions is due to Fishburn (1981) who showed that there always exist a preference profile for which removing any candidate from A leads to the reversed ranking on the remaining set of candidates. In a seminal paper, Saari (1988) generalized this result by studying simultaneously the rankings on all the subsets of A for a given profile. He showed that, for most of the scoring rules (the Borda rule being one of the few exceptions) anything can happen for some profiles and no relationship prevails between the rankings on different subsets. This result was further developed in Saari (1987, 1990, 1996).

This result could cast a doubt on the practical use of scoring rules. What remains is to see whether these paradoxical results are just rare oddities or betray a more generalized behavior. In modern social choice theory, many works have tried to analyze the relationships between pairwise and scoring rules. Among others, we can mention the works of Gehrlein and Fishburn (1976, 1980, 1981a), Gehrlein et al. (1982), Fishburn (1981), Tataru and Merlin (1997), Van Newenhizen (1992) and more recently, Cervone et al. (2005). Most of them prove that given a scoring rule and a collective ranking, there is no reason to think that the pairwise comparisons will always be consistent with this collective ranking.

However, this line of research barely analyzed anything but pairwise relationships. In three-candidate elections, Gehrlein and Fishburn (1980, 1981a) computed the limit probabilities under the Impartial Culture (IC) condition (defined later) that, given a scoring rule, the pairwise comparisons between candidates agree with the collective ranking. They showed that, the agreement is maximized by the Borda rule and is minimized[6] by the Plurality rule and the Antiplurality rule.

Only one reference dealt with the relationship between the four-candidate set and the three-candidate subsets: Gehrlein and Fishburn (1980). First, they established that, the probability of agreement between pairwise comparisons and the collective ranking is maximized by the Borda rule. Next, they computed the mean limit probability that the collective ranking on three candidates agree with the collective ranking on four candidates. One drawback of their approach is that, they evaluated the likelihood of the same ranking on $\{a, b, c, d\}$ and $\{a, b, c\}$ regardless of the position of d in the four-candidate ranking; they only cared about the possibility of lifting up the initial ranking on $\{a, b, c\}$ to the superset.

Our objective in this paper will be to derive the probability of any ranking on the three-candidate subset given that any candidate has been removed from the four-candidate set.[7] In this paper, we enrich Gehrlein and Fishburn (1980)'s analysis (1) *by obtaining exact probabilities of consistency depending on the original position of the removed candidate*; (2) *by deriving in each case, the likelihood of all the possible rankings on subsets*. Our probability computations not only lead us to make an hierarchy of the main scoring rules according to their stability; *we also derive from them, the likelihood of the two electoral paradoxes for committee elections presented above*.

The rest of the paper is structured as follows: Section 2 is devoted to basic notations and definitions. In Section 3, we motivate the paper by considering some examples showing how the collective rankings over proper subsets of a set of alternatives can be consistent or not with the collective ranking of this set. In Section 4, we evaluate this event in the four-alternatives case using the impartial culture condition. In Section 5, we provide the formal definitions of the PSP and the LMP and we then derive their likelihood in four-candidate elections and two-member committees as a consequence of our probability computations.

2. Notation and definitions

2.1. Preferences

Let N be the set of n voters ($n \geq 2$) and A the set of m alternatives or candidates, $m \geq 3$. The binary relation R over A is a subset of the cartesian product $A \times A$. For $a, b \in A$, if $(a, b) \in R$, we write aRb to say that "a is at least good as b". $\neg aRb$ is the negation of aRb. If we have aRb and $\neg bRa$, we will say that "a is better or strictly preferred to b". In this case, we write aPb with P the asymmetric component of R. The preference profile $\pi = (P_1, P_2, \ldots, P_i, \ldots, P_n)$ gives all the linear orders[8] of all the n voters on A where P_i is the strict ranking of a given voter i over A. When we consider the preference

[3] See Young (1988) for a modern interpretation of Condorcet's rule.

[4] Let us take three candidates a, b and c in order to illustrate what a majority cycle is in a simple way. For a given electorate, if a is majority preferred to b and b is majority preferred to c and c is majority preferred to a, this describes a majority cycle with three candidates.

[5] Eventually, the Borda rule was retained by the Academy.

[6] Gehrlein and Fishburn (1980, 1981a) showed that with three candidates a, b, c, given that the collective ranking is abc, the limit probability to have a majority preferred to b (or b majority preferred to c) is 85.3% for the Borda rule and 75.5% for the Plurality rule and the Antiplurality rule; the limit probability to have a majority preferred to c is 96.9% for the Borda rule and 90.1% for the Plurality rule and the Antiplurality rule.

[7] Notice that, in Gehrlein and Fishburn (1980, 1981a), Fishburn (1981) and Saari (1987), Saari (1988, 1990, 1996) as it will be the case in this paper, when a candidate is removed, it will not be for strategic or for manipulation purpose as in Tideman (1987) or Dutta et al. (2001). Removing a candidate is strategic if the objective is to improve the ranking of a certain candidate. We will also show that, our results hold for situations in which a new candidate is added in a three-candidate elections. Results on related issue have been recently discovered by Chevaleyre et al. (2012), Lang et al. (2013). We are not going to say more about this point since we are not concerned with the strategic aspect of the withdrawing of one or more candidates.

[8] A linear order is a binary relation that is transitive, complete and antisymmetric. The binary relation R on A is *transitive* if for $a, b, c \in A$, if aRb and bRc then aRc. R is *antisymmetric* if for all for $a \neq b$, $aRb \Rightarrow \neg bRa$; if we have aRb and bRa, then $a = b$. R is *complete* if and only if for all $a, b \in A$, we have aRb or bRa.