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Manipulation in games with multiple levels of output

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ABSTRACT

In (j, k)-games each player chooses amongst j ordered options and there are k possible outcomes. In this paper, we consider the case where players are assumed to prefer some outcomes to others, and note that when $k > 2$ the players have an incentive to vote strategically. In doing so, we combine the theory of cooperative game theory with social choice theory, especially the theory of single-peaked preferences. We define the concept of a (j, k)-game with preferences and what it means for it to be manipulable by a player. We also consider Nash equilibriums with pure strategies for these games and find conditions that guarantee their existence.

1. Introduction

Simple games, in which a set of voters have two possible votes and there are two possible outputs, have been studied in the voting context since von Neumann and Morgenstern (1944). More recently in Fishburn (1973), Rubinstein (1980), and Felsenthal and Machover (1997) these games are extended to allow voters three ordered possible options and two outputs. In Freixas and Zwicker (2003) this idea is generalized to include many ordered levels of approval for the voter and many ordered levels of output. Games with multiple levels of input and output, and in particular how relative power can be determined amongst the players, have also been studied in Parker (2012) and Pongou et al. (2011). So far, the study of these games has always assumed that the players are voting in a genuine fashion and accept the final outcome. However, when the number of outputs is greater than two and a player prefers one of the middle outcomes, then she may wish to alter her vote from the one that most closely represents her honest input to another simply to have better chances to reach an outcome being as close as possible to her most preferred one. Or to put it in a way that is more charitable to the voter, the game itself fails to provide the voter a vote that accurately represents her preferences in all cases. In Section 1.1, we give natural examples of voting situations in the areas of: law, academics, appropriations, and politics where such situations arise.

There are of course many results in the field of social choice theory dealing with the situation where there is an election with three or more candidates where manipulation by the voters is possible, of which Arrow (1950) is the most famous, see for example Taylor and Pacelli (2008). However, this situation is different from the most general cases in social choice theory since the outcomes are assumed to have a natural ordering and this ordering is agreed upon by all voters. Thus, although a voter may prefer any outcome as their most desired outcome, their other preferences must be ordered in a way that is consistent with the overall ordering. This notion is made rigorous in Section 2. In addition, this problem is distinguished from many social choice situations (e.g., social choice procedures or social welfare functions) since in our context the input to the voting system is a single vote, which is not necessarily an element of the set of output alternatives, rather than an ordered list of output alternatives.

The paper is organized as follows. Some motivating examples are given in the rest of Section 1. Section 2 begins with the formalizations and terminologies of the notions of (j, k)-games with preferences and that of manipulation for this type of games. Section 3 begins with a comparison of how the results in Gibbard (1973) would apply if either we allowed players to have arbitrary preferences or we do not. The section also includes some basic results on manipulation for (j, k)-games with preferences. In Section 4 we will turn our attention to looking at Nash equilibrium with pure strategies. Our starting point is the consideration of some subclasses of games with preferences having Nash equilibrium.

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We continue by showing the existence of games with preferences with as few as three players without Nash equilibrium. The central question in Section 5 is whether Nash equilibrium with pure strategies exists for anonymous games with preferences. Although we leave the general question as an open conjecture, we partially prove it for some particular subcases. In Section 6 we conclude the paper.

1.1. Examples

These examples have a common thread: The individual voters have a finite and naturally ordered set of voting options to choose from, and the body as a whole has a finite and naturally ordered set of outcomes. Thus can easily be modeled as a (j, k)-game from Freixas and Zwicker (2003). Similar examples have been explored in continuous framework of spatial and directional see Enelow and Hinich (1990) and Rabinowitz and Macdonald (1989) respectively. But also, these examples have in common that the system is set up under the assumption that each individual is not motivated by hoping for a particular outcome, but rather voting in a way that is their best assessment of what the best option is and trusting the voting system to produce the best outcome. However, if the voter is more interested in the final outcome than in the honesty of his or her actual vote, then as these examples show, the voter may not have a vote that will correspond to giving the best chance that his or her top preference will be the final outcome. Instead, the voter must guess or investigate how others will vote in order to decide on his or her vote. From these examples we can see many of the problems from the literature of social choice theory will also occur in the theory of (j, k)-games.

Example 1. A juror has two choices – convict or acquit – but the outcome of the jury as a whole has a third option: a hung jury. For purposes of the example, we will suppose there are 12 jurors and each will vote for either conviction or acquittal and the outcome of the vote will be the unanimous decision of the jury, with a hung jury (and hence a potential retrial) if unanimity is not achieved. In reality, this vote might be taken many times in the process, with time for the jurors to persuade others between votes. We will assume we are just looking at the last vote. Hence if unanimity fails the jury will be hung.

The system is certainly set up under the assumption that each juror is supposed to give their honest assessment of the evidence. However, it is certainly plausible that a juror will prefer a hung jury, believing the proceedings were unfair and hence there should be a retrial. Although it is the job of the judges and not jury to decide the fairness of the trial, we must realize that individual motivations are often different from what the system assumes.

In this case, the juror will have to vote based on her assumption of what the other votes will be. She has no vote available to her that is the "best" in any objective sense.

Example 2. At a certain school each student is given a grade in each class of A, B, C, D or F. A student graduates if he passes every course (with a D or better) and has a grade point average of a C average or better. Each of his teachers can then be thought of as voters deciding if he should graduate. A teacher might think that he deserves a D in her class but also wants him to graduate. She would like to give him the lowest grade possible without costing him his graduation.

Example 3. Suppose each member of a budget committee is asked to vote to allocate either 5, 10, 15 or 20% of the budget to fund a project. The final allotment will be the average of the members' suggestions. Again, if a committee member is more interested in the outcome of the averaging than her recommendation corresponding to her honest assessment, then she may wish to adjust her recommendation based on what she anticipates will be the recommendation of the others. In the cases where her honest assessment is that either 5 or 20% of the budget should be spent on the project, then she has no decision to make. Her best strategy and her honest assessment are identical and she does not have to take into account how others might vote. Only when she wants a middle value, does she have to consider strategizing.

A variation of this example would be the case in which the final allotment will be the median of the members' suggestions. Again, if a committee member is more interested in the outcome, she can radicalize her vote to the extreme option, either 5 or 20% of the budget. However, the chance to manipulate, especially if the number of voters is not small, is here lower than for the mean.

So far in all of the examples the voting has been symmetric amongst all voters, that is, each player's vote is treated the same. In such a framework, names need not be attached to the votes, so these games are called anonymous. The framework of (j, k)-games is certainly flexible to allow for the voters to have different roles. We can modify the previous example to allow for asymmetric voting.

Example 4. Suppose in the above example there is a budgetary supervisor who determines the maximum amount that can be spent on a project. Hence, he also votes 5, 10, 15 or 20%, but his vote represents the maximum amount that can be spent on the project. Thus the outcome of the voting will be the smaller of the average of all the non-supervisors' votes and the vote of the supervisor. We can see that all of the non-supervisors could have an incentive to vote strategically, but it is not clear the supervisor does.

These examples include the possibility that a player may vote in a way to increase her chance to get her favorite outcome, and to do so she might have to guess what other voters are going to do. That is, she may not have a vote that best represents her preferences independent of the actions of all other players. In the paper we will call such a game "manipulable". This term does have a negative connotation and seems to imply that a player changing her vote based on trying to get her preferred outcome is somehow doing something unethical. It has been suggested that a better term for this kind of voting would be either "strategic", "tactical" (more positive) or "insincere" (more negative) voting and reserve the term "manipulation" for more untoward activities such as bribing voters or destroying ballots. We do not mean to imply any moral judgement upon a player using her best strategy. If the term manipulable is pejorative it is against the game not providing the player a vote that best represents her preferences.

2. Definitions and notation

The following two definitions are from Freixas and Zwicker (2003) while adopting some of the notation from Felsenthal and Machover (1997):

Definition 1. An ordered j-partition of a finite set N of players or voters is a sequence $A = (A_1, A_2, \ldots, A_j)$ of disjoint, possibly empty sets whose union is N. If $a \in A_i$ we say that a approves at level i or votes for the ith option. If $i_1 < i_2$ we say those voting at approval level i_1 are voting at a higher level of approval than those approving at level i_2. We denote the set of all j-partitions of N by j^N. For every ordered j-partition A we define $T_A : N \to \{1, 2, \ldots, j\}$ by $T_A(a) = i$ if $a \in A_i$. For two ordered j-partitions A and B, we write $A^j \subseteq B$ if $T_A(a) \geq T_B(a)$ for all $a \in N$. We say the ordered j-partitions, A and B *agree outside of player a* if $T_A(x) = T_B(x)$ for all $x \neq a$.