Journal of Mathematical Economics 61 (2015) 157–165

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

A nonsmooth approach to envelope theorems[☆]

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ARTICLE INFO

Article history:
Received 2 April 2012
Received in revised form
31 August 2015
Accepted 2 September 2015
Available online 11 September 2015

Keywords:
Constrained otimization with nonconvexities
Envelope theorems
Nonsmooth analysis
Stochastic growth
Lattice programming

ABSTRACT

We develop a nonsmooth approach to envelope theorems applicable to a broad class of parameterized constrained nonlinear optimization problems that arise typically in economic applications with nonconvexities and/or nonsmooth objectives. Our methods emphasize the role of the Strict Mangasarian–Fromovitz Constraint Qualification (SMFCQ), and include envelope theorems for both the convex and nonconvex case, allow for noninterior solutions as well as equality and inequality constraints. We give new sufficient conditions for the value function to be directionally differentiable, as well as continuously differentiable. We apply our results to stochastic growth models with Markov shocks and constrained lattice programming problems.

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1. Introduction

Since the work of Viner (1931) and Samuelson (1947), the envelope theorem has become a standard tool in economic analysis. In its "classical" form an envelope theorem is simply a continuous derivative of the value function in a parameter. Sufficient conditions for its existence at first required a great deal of mathematical structure, including convexity, interiority, as well as the continuous differentiability ("smoothness") of objectives and constraints (e.g., Samuelson, 1947, Rockafellar, 1970, Mirman and Zilcha, 1975, Benveniste and Scheinkman, 1979).

In subsequent work, some of these assumptions were loosened. For instance, versions of Danskin's Theorem in Clarke (1975) and Milgrom and Segal (2002) relax conditions on the structure of the choice set in unconstrained problems, with Clausen and Strub (2013) further extending these results to problems with interior solutions and integer decisions in very general dynamic settings.

Recently, and more along the lines of this paper, Rincon-Zapatero and Santos (2009) have extended the classical C^1 envelope theorem to infinite horizon stochastic dynamic programs with inequality constraints in the presence of noninterior solutions. These findings (as well as Milgrom and Segal's results for the cases with inequality constraints), however, only concern convex programs and it is not clear how they can be extended to economic models with nonconvexities and/or non-differentiable objectives. At the same time, the optimization literature has made a lot of progress on stability bounds for nonconvex non-smooth programs (see, for instance, the monographs of Clarke, 1983 and Bonnans and Shapiro, 2000), although the focus has generally not been on simple and practical sufficient conditions for exact directional derivatives.

In this paper we combine recent results from the optimization literature with sets of conditions easily verifiable in finite dimensional problems sufficient for the existence of generalized envelope theorems applicable to many economic models with nonconvexities or non-smooth objectives. When seeking envelope theorems for such programs, several important issues arise. First, since classical envelope theorems cannot be expected, one would like to propose an alternative notion of a "generalized" envelope that fits most applications and is a "substitute" for the classical envelope. Second, the proposed approach must work in settings with

☆ We thank Rabah Amir, Bob Becker, Madhav Chandrasekher, Andrew Clausen, Bernard Cornet, Manjira Datta, Amanda Friedenberg, Karl Hinderer, Felix Kubler, Rida Laraki, Cuong LeVan, Len Mirman, Andrzej Nowak, Ed Prescott, Manuel Santos, Carlo Strub, Yiannis Vailakis, and Lukasz Woźny, and especially Atsushi Kajii, and two referees for very helpful discussions and comments during the writing of this paper. Kevin Reffett thanks the Centre d'Economie de la Sorbonne (CES) and the Paris School of Economics for arranging his visits during the Spring terms of both 2011 and 2012. The usual cavaets apply.

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http://dx.doi.org/10.1016/j.jmateco.2015.09.001
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both equality and inequality constraints, and when optimal solutions are not necessarily interior. Third, when Slater's condition is not appropriate, new constraint qualifications allowing for simple (and, if possible, exact) calculations of a generalized envelope or for the existence of differential bounds for the value function (the latter is often all that is needed in applications) need to be identified.

Methodologically, we take a "nonsmooth" approach closely related to the work of Gauvin and Tolle (1977), Gauvin and Dubeau (1982) and Auslender (1979), and consider Lipschitz programs in finite dimensional spaces, in which objective functions are only assumed to be locally Lipschitz, thus not necessarily differentiable, and the constraints are continuously differentiable.[1] We show how progressively stronger conditions on primitive data lead to progressively sharper characterizations of the differentiable properties of the value function. Specifically, we give conditions under which value functions admit differential bounds, are Clarke differentiable, directionality differentiable, and continuously differentiable. Our sharpest results focus on constraint systems satisfying the Strict Mangasarian–Fromovitz Constraint Qualification, a refinement of the Mangasarian–Fromovitz Constraint Qualification equivalent to the uniqueness of the Karush–Kuhn–Tucker multiplier in our setup.

The remainder of the paper is laid out as follows. In Section 2, we describe the benchmark class of optimization programs we consider. In Section 3 we present our main results. In particular, we provide results on differential bounds for the value function, directional differentiability, and C^1 differentiability. Applications of some of these results make up the last section of the paper, and the Appendix briefly exposes some mathematical tools of nonsmooth analysis and some lattice programming notions.

2. Lipschitz programs

Given a space A of actions (or controls), a parameter space S, and an objective function $f : A \times S \to \mathbb{R}$, we consider the following *parameterized Lipschitz program*:

$$V(s) = \max_{a \in D(s)} f(a, s) \tag{2.0.1}$$

where the feasible correspondence is given by:

$$D(s) = \{a | g^i(a, s) \le 0, \ \ i = 1, \dots, p, \ h^j(a, s) = 0, \ j = 1, \dots, q\}$$

and the optimal solution correspondence is defined as:

$$A^*(s) = \arg\max_{a \in D(s)} f(a, s).$$

We will maintain some baseline assumptions throughout the paper:

Assumption 1. (a) A and S are open convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively;
(b) the objective function $f : A \times S \to \mathbb{R}$ is locally Lipschitz in (a, s),
(c) the constraints $g^i, i = 1, \dots, p$ and $h^j, \ j = 1, \dots, q$ are jointly C^1, and $n \ge q$.

We use a standard Lagrangian duality scheme applied in a nonconvex context. Of course, the cost of relaxing convexity is immediate as we generally lose strong duality and the construction of a necessary and sufficient first order theory for optimal solutions generically becomes impossible. Nevertheless, this does not prevent us from providing mild conditions under which sharp characterizations of simple nonsmooth envelope theorems for program (2.0.1) can be read from linearizations of the Lagrangian dual at its optimum.

Letting the vector of dual variables be denoted by $(\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q$, we conjugate program (2.0.1) with a classical Lagrangian duality scheme as follows[2]:

$$L(a, \lambda, \mu; s) = \begin{cases} f(a, s) - \lambda g(a, s) - \mu h(a, s) & \text{if } a \in A, (\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q \\ -\infty, & \text{if } a \notin A, (\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q \\ \infty, & \text{otherwise} \end{cases}$$

where:

$$g(a, s) = [g^1(a, s), \dots, g^p(a, s)];$$
$$h(a, s) = [h^1(a, s), \dots, h^q(a, s)].$$

A point $a \in D(s)$ is a *Karush–Kuhn–Tucker (KKT) point* if there exists $(\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q$ such that:

$$0 \in \partial_a f(a, s) - (\lambda \nabla_a g + \mu \nabla_a h)(a, s)$$

and:

$$\lambda g(a, s) = 0.$$

We denote by $K(a, s)$ the set of "KKT multipliers" (λ, μ) associated with the KKT point a.

In constrained optimization, constraint qualifications are needed to guarantee that optimal solutions are KKT points, and that the feasible region around such points does not vanish under local perturbations of the parameters. The strongest of these constraints is the Linear Independence Constraint Qualification (LICQ), central to the work of Gauvin and Dubeau (1982) and the focus of Rincon-Zapatero and Santos (2009), for example.

Definition 1. A feasible point $a \in D(s)$ satisfies the LICQ if the following vectors are linearly independent,

$$\nabla_a g^i(a, s), i \in I, \qquad \nabla_a h^j(a, s), j = 1, \dots, q$$

where $I = \{i : g^i(a, s) = 0\}$.

The LICQ is rather strong and plays no role in our argument. Rather, we focus a weaker constraint, the Mangasarian–Fromovitz Constraint Qualification (MFCQ), which is equivalent to Robinson's constraint qualification in finite dimensional spaces (and with a finite number of constraints).

Definition 2. A feasible point $a \in D(s)$ satisfies the MFCQ if:

(i) the following vectors are linearly independent,
$$\nabla_a h^j(a, s), \quad j = 1, \dots, q$$
(ii) there exists $y \in \mathbb{R}^n$ such that,
$$\nabla_a g^i(a, s)y < 0, \quad i \in I; \qquad \nabla_a h^j(a, s)y = 0, \quad j = 1, \dots, q$$
where the set of active constraints are denoted by $I = \{i : g^i(a, s) = 0\}$.

Because the MFCQ is not sufficient to guarantee the uniqueness of the KKT multipliers, it is very difficult to obtain directional derivatives and sharp characterizations of the generalized gradient. However, the MFCQ is sufficient for asserting the nonemptiness of the set of multipliers (as well as their compactness; e.g., Gauvin and Tolle, 1977, Corollary 2.8), as stated in the following proposition.

[1] Problems with nonsmooth constraints as well as "mixed-integer programming" problems are studied in Morand et al. (2013).

[2] If A is closed, then the abstract constraint $a \in A$ induces an additional term in the Lagrangian (see, for instance, Clarke, 1983, Chapter 6).

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