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Multiplication and comultiplication of beliefs

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Abstract

Multiplication and comultiplication of beliefs represent a generalisation of multiplication and comultiplication of probabilities as well as of binary logic AND and OR. Our approach follows that of subjective logic, where belief functions are expressed as opinions that are interpreted as being equivalent to β probability distributions. We compare different types of opinion product and coproduct, and show that they represent very good approximations of the analytical product and coproduct of β probability distributions. We also define division and codivision of opinions, and compare our framework with other logic frameworks for combining uncertain propositions.

Keywords: Belief calculus; Subjective logic; Dempster–Shafer; Belief theory; Conjunction; Disjunction; Unconjunction; Undisjunction; Three valued logic; Support logic

1. Introduction

Subjective logic [2] is a belief calculus based on the Dempster–Shafer belief theory [5]. In subjective logic the term *opinion* denotes beliefs about propositions, and a set of standard and non-standard logic operators can be used to combine opinions about propositions in various ways. A particular type of multiplication and comultiplication called *propositional conjunction* and *propositional disjunction* in [2] will be called *simple multiplication* and *simple comultiplication* here. In [2] it was also

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described how every opinion can be uniquely mapped to a β probability distribution, thereby providing a specific interpretation of belief functions in Bayesian probabilistic terms. A vacuous opinion about a binary proposition is for example equivalent to a uniform probability distribution. A question left open in [2] was why simple multiplication of two vacuous opinions produces a product opinion that when mapped to a β distribution is slightly different from the analytical product of two uniform distributions. Below we will explain the reason for this difference, and also define alternatives to simple multiplication and simple comultiplication in the form of *normal multiplication* and *normal comultiplication* of opinions. Simple and normal multiplication and comultiplication are compared to analytical multiplication and comultiplication of β distributions in the general case. We also define the inverse opinion operators *normal division* and *normal codivision*.

2. Fundamentals of subjective logic

Subjective logic is suitable for approximate reasoning in situations where there is more or less uncertainty about whether a given proposition is true or false, and this uncertainty can be expressed by a belief mass assignment [1] (BMA) where a quantity of belief mass on a given proposition can be interpreted as contributing to the probability that the proposition is true. More specifically, if a set denoted by Θ of exhaustive mutually exclusive singletons can be defined, this set is referred to as a frame of discernment. Each singleton, which will be called an *atomic element* hereafter, can be interpreted as a proposition that can be either true or false. The powerset of Θ denoted by 2^Θ contains all possible subsets of Θ. The set $2^\Theta - \{\emptyset\}$ of nonempty subsets of Θ will be called its reduced powerset. A BMA assigns belief mass to nonempty subsets of Θ (i.e. to elements of $2^\Theta - \{\emptyset\}$) without specifying any detail of how to distribute the belief mass amongst the elements of a particular subset. In this case, then for any non-atomic subset of Θ, a belief mass on that subset expresses uncertainty regarding the probability distribution over the elements of the subset. More generally, a belief mass assignment m on Θ is defined as a function from $2^\Theta - \{\emptyset\}$ to $[0, 1]$ satisfying:

$$\sum_{x \subseteq \Theta} m(x) = 1. \tag{1}$$

Each nonempty subset $x \subseteq \Theta$ such that $m(x) > 0$ is called a focal element of m. Special names are used to describe specific BMA classes. When $m(\Theta) = 1$ the BMA is *vacuous*. When all the focal elements are atomic elements, the BMA is Bayesian. When $m(\Theta) = 0$ the BMA is *dogmatic* [7]. Let us note, that trivially, every Bayesian belief function is dogmatic. When all the focal elements are nestable (i.e. linearly ordered by inclusion), then the BMA is *consonant*.

[1] Called *basic probability assignment* in [5].