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# Equilibria in a class of aggregative location games

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ABSTRACT

Consider a multimarket oligopoly, where firms have a single license that allows them to supply exactly one market out of a given set of markets. How does the restriction to supply only one market influence the existence of equilibria in the game? To answer this question, we study a general class of aggregative location games where a strategy of a player is to choose simultaneously both a location out of a finite set and a non-negative quantity out of a compact interval. The utility of each player is assumed to depend solely on the chosen location, the chosen quantity, and the aggregated quantity of all other players on the chosen location. We show that each game in this class possesses a pure Nash equilibrium whenever the players' utility functions satisfy the assumptions *negative externality, decreasing marginal utility, continuity,* and *Location–Symmetry*. We also provide examples exhibiting that, if one of the assumptions is violated, a pure Nash equilibrium may fail to exist.



## 1. Introduction

We introduce a class of aggregative games that combines the characteristics of finite games, such as congestion games, and continuous games, such as Cournot oligopolies. As an example of a game in this class, consider a multimarket oligopoly in which each firm may offer a positive quantity on exactly one market only, but is free to choose the market out of a set of feasible markets. This situation arises, for instance, if governmental policies oblige each firm to be engaged in at most one market at a time, e.g., by issuing only a single license per firm. Mathematically, the restriction on a single positive quantity renders the strategy space to be non-convex. As a consequence, standard tools, such as fixed point theorems à la Kakutani, are not directly applicable to establish the existence of equilibria.

In this paper, we consider a general class of aggregative games that includes multimarket oligopolies with licenses discussed above as a special case. Formally, let $A$ be a finite set of locations and $N = \{1, \dots, n\}$ be a finite set of players. Each player is associated with a non-empty subset $A_i \subseteq A$ of feasible locations and a non-empty and compact interval of non-negative quantities $Q_i$ feasible to her. In a strategy profile, each player $i$ chooses simultaneously both a feasible location $a_i \in A_i$ and a feasible quantity $q_i \in Q_i$. We require that the utility of each player depends solely on the location chosen, her own quantity, and the aggregated quantity of all other players choosing the same location. It is a useful observation that for such an aggregative game the utility of each player $i$ in strategy profile $(a, q) = (a_1, \dots, a_n, q_1, \dots, q_n)$ can be represented by a set of indirect utility functions $v_{i,\sigma} : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}$, $i \in N$, $\sigma \in A_i$ so that $u_i(a, q) = v_{i,a_i}(q_i, \ell_{-i,a_i}(a, q))$, where, for an arbitrary location $\sigma \in A$, we denote by $\ell_{-i,\sigma}(a, q) = \sum_{j \in N \setminus \{i\}: a_j = \sigma} q_j$ the aggregated quantity of all players except $i$ on location $\sigma$.

We impose the following four assumptions on the player's indirect utility functions. The first assumption, called "Negative Externality", requires that the indirect utility of a player does not increase if the aggregated quantity of the other players on the same location increases. Informally, the second assumption "Decreasing Marginal Utility" requires that, for every player, the marginal indirect utility function exists and decreases if both the player's quantity and the aggregated quantity of the chosen location increase.[1] Third, we require that the indirect utility functions of each player are continuous. The last assumption is called "Location–Symmetry" and requires that, for each player $i$, we have $v_{i,\sigma} = v_{i,\tau}$ for all $\sigma, \tau \in A_i$.

We prove that aggregative games for which the indirect utility functions satisfy "Negative Externality", "Decreasing Marginal

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[1] As a consequence of this assumption, players will lower their quantities when their competitors raise their quantities. Thus, this assumption can also be seen as a variant of "strategic substitutes" (Bulow et al., 1985; Dubey et al., 2006).

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Utility", "Continuity", and "Location–Symmetry" possess a pure Nash equilibrium. To prove this existence result, we devise an algorithm that computes a pure Nash equilibrium. Our algorithm relies on iteratively computing a *partial* equilibrium on every location separately using Kakutani's fixed point theorem. Here, a partial equilibrium is a strategy profile that is resilient against unilateral *quantity* deviations. Given a partial equilibrium, the algorithm selects a single player who can strictly improve and computes for this player a best reply. After such a best reply it recomputes a partial equilibrium and reiterates. We prove that a player-specific aggregated quantity vector of the partial equilibria lexicographically decreases in every iteration and, thus, the algorithm terminates after a finite number of iterations. A perhaps surprising property of our proof is that even though we iteratively recompute a partial equilibrium – using Kakutani's theorem as a black box – there is enough structure of such a partial equilibrium to prove that the algorithm terminates. For games with only two players, we prove that the assumption "Location–Symmetry" is not needed to guarantee the existence of an equilibrium, i.e., already "Negative Externality", "Decreasing Marginal Utility", and "Continuity" of the players' indirect utilities are sufficient to yield the existence of a pure Nash equilibrium. To demonstrate the usefulness of our results, we give concrete examples of games that fit into our model: restricted multimarket oligopolies, congestion games with variable quantities, and multiserver queuing games. For all these examples, to the best of our knowledge, we establish for the first time the existence of a pure Nash equilibrium.

### 1.1. Related work

Many works on the existence of Nash equilibria in strategic games impose strong assumptions on the topological properties of the players' strategy sets. Most prominently, Nash's famous existence result for equilibria in mixed extensions of finite games uses the fact that the mixed strategy set of each player is the simplex spanned by her pure strategies and, thus, a well-behaved convex and compact subset of some Euclidean space. The existence of a mixed equilibrium is then established via the fixed point theorems of Kakutani (cf. Nash, 1950a) or Brouwer (cf. Nash, 1950b).[2] For these fixed point arguments, however, the convexity of the (mixed) strategy sets is crucial. We are interested in pure Nash equilibria in this work, and, for our class of games, the strategy space of a player is not necessarily convex.

A strategically equivalent game to ours with convex strategy space can be obtained by taking the convex hull of the strategy space and assigning sufficiently low utility values for infeasible strategies. This method inevitably leads to games with *discontinuous* utility functions. There is a substantial body of literature on discontinuous games that identifies conditions under which an equilibrium exists (cf. Barelli and Meneghel, 2013, Bich, 2009, Carmona, 2009, 2011, Dasgupta and Maskin, 1986, McLennan et al., 2011, Reny, 1999 and Simon, 1987). To the best of our knowledge, the currently most general sufficient condition for the existence of an equilibrium is given by Barelli and Meneghel (2013) using the concept of *continuous security*. In Appendix B.2, we show that this condition is not satisfied for our class of games. It is also straightforward to show that our games are not supermodular (see Appendix B.3) which prevents us from the application of Tarski's fixed-point theorem or further comparative statics analysis (cf. Amir, 1996, Milgrom and Roberts, 1990; Milgrom and Shannon, 1994, Roy and Sabarwal, 2010, Tarski, 1955, Topkis, 1979, 1998 and Vives, 1990 for works in this field).

Closer to our work, Dubey et al. (2006) considered a class of games, for which the strategy set of each player is a compact and possibly non-convex subset of the non-negative real line, and the utility of each player depends only on her own strategy, and the sum of the others' strategies. They derived the existence of a pure Nash equilibrium assuming that there exists a selection from the best reply correspondence of each player, which is non-increasing or non-decreasing in the aggregated strategy of the other players.[3] This assumption is met, e.g., by Cournot oligopolies. Jensen (2010) generalized the work of Dubey et al. and Kukushkin as he allows for higher dimensional strategy sets. In Jensen's model, the utility of each player only depends on her own strategy and a one-dimensional aggregate of the strategies of the other players. In particular, the aggregate is independent from the own strategy. This is in contrast to our model where the utility of a player depends on the chosen location, her own quantity, and the aggregated quantity of all other players that choose the same location.

Very recently, Martimort and Stole (2011) presented several characterizations of equilibria in aggregative games which lead, however, not directly to existence results. Only for special cases (such as affine utilities) they establish sufficient conditions for the existence of an equilibrium.

An extended abstract of this paper appeared in the Proceedings of the 5th Workshop on Internet and Networks Economics (Harks and Klimm, 2011a).

### 1.2. Paper outline

The remainder of this paper is organized as follows. In Section 2, we introduce our basic model of an aggregative game and the assumptions that we impose on the players' utility functions. In Sections 3.1 and 3.2, we provide our main results for the existence of a pure Nash equilibrium in games with an arbitrary number of players and two-player games, respectively. In Section 3.3, we discuss how the assumptions "Continuity" (CON) and "Location–Symmetry" (LOC) can be weakened. In Section 4, we demonstrate the usefulness of our results by giving several applications that fit into our model. In Appendix A, we complement our results and show that if one of our assumptions on the indirect utility functions is violated, then there is a game without a pure Nash equilibrium. In Appendix B, we discuss the relationship of our existence result to known results for potential games, discontinuous games, and supermodular games.

## 2. The model

Let $A$ be a finite set of locations and let $N = \{1, \dots, n\}$ be a finite set of players. For each player $i \in N$ we are given a closed interval $Q_i = [\alpha_i, \omega_i] \subseteq \mathbb{R}_{\geqslant 0}$ of feasible quantities and a subset $A_i \subseteq A$ of feasible locations. A *strategy* of player $i$ is a tuple $(a_i, q_i)$ where $a_i \in A_i$ is a feasible location and $q_i \in Q_i$ is a feasible quantity for player $i$. A *strategy profile* of the game is a tuple $(a, q)$ where $a = (a_1, \dots, a_n)$ is the location profile and $q = (q_1, \dots, q_n)$ is the quantity profile. Note that for $|A_i| > 1$, embedding $A_i$ into $\mathbb{N}$ renders the strategy space $S_i = A_i \times Q_i$ into a non-convex subset of $\mathbb{R}^2_{\geqslant 0}$. For a location $\sigma \in A$, and a strategy profile $(a, q)$, we denote by $\ell_\sigma(a, q) = \sum_{j \in N: a_j = \sigma} q_j$ the aggregated quantity on location $\sigma$

[2] Further generalizations of Kakutani's fixed point theorem to strategy spaces that are non-empty, convex and compact subsets of Hausdorff locally convex topological vector spaces can be found in Debreu (1952), Fan (1952), and Glicksberg (1952).

[3] See also Kukushkin (1994, 2004) for related results on the existence of equilibria and the convergence of improvement dynamics in finite non-convex games satisfying strategic complementarities and substitutes.