Journal of Mathematical Economics 55 (2014) 4–10

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Cooperative equilibria of finite games with incomplete information

Mitsunori Noguchi

Meijo University, Japan

ARTICLE INFO

Article history:
Received 7 April 2014
Received in revised form
26 August 2014
Accepted 14 September 2014
Available online 20 September 2014

Keywords:
Alpha-core
Game with incomplete information
Pure strategies
Behavioral strategies
Young measure

ABSTRACT

Recently, Askoura et al. (2013) proved the nonemptiness of the α-core of a finite Bayesian game $G_{\mathcal{R}}$ with Young measure strategies and nonatomic type spaces, without requiring that the expected payoffs be concave. Under the same hypotheses as theirs, we demonstrate that Scarf's method (1971) works with some adjustments to prove the nonemptiness of the α-core of a similar game $G_{\mathcal{M}}$ with pure strategies. We prove that the nonemptiness of the α-core of a $G_{\mathcal{M}}$ is equivalent to that of its associated characteristic form game $G^C_{\mathcal{M}}$, that the core of $G^C_{\mathcal{M}}$ and hence the α-core of a $G_{\mathcal{M}}$ is nonempty, and that the nonemptiness of the α-core of a $G_{\mathcal{M}}$ is equivalent to that of a $G_{\mathcal{R}}$, which clearly implies the result of Askoura et al. (2013). Our proofs hinge on an iterated version of Lyapunov's theorem for Young measures to purify partially as well as fully Young measure strategies in an expected payoff function, which is a main methodological contribution of this paper.

1. Introduction

Recently, Askoura et al. (2013) proved the nonemptiness of the α-core of a finite game with incomplete information $G_{\mathcal{R}}$, in which players choose behavioral (Young measure) strategies δ, without requiring that the expected payoffs be concave in δ; they require, however, that the payoff functions themselves be concave in actions, that the individual type spaces be non-atomic, and that the common prior governing the type profiles be absolutely continuous with respect to the product of its marginals. The present author was inspired by their work, which is novel in that it identifies a class of finite games with incomplete information, non-concave payoffs, and nonempty α-core.

In this paper we demonstrate that under the same hypotheses as those in Askoura et al. (2013), Scarf's method (1971) can be utilized after necessary adjustments to prove the nonemptiness of the α-core of a finite game with incomplete information $G_{\mathcal{M}}$, in which players choose pure strategies instead of behavioral strategies.

To be more specific, we prove that the nonemptiness of the α-core of a $G_{\mathcal{M}}$ is equivalent to that of its associated characteristic form game $G^C_{\mathcal{M}}$, that the core of $G^C_{\mathcal{M}}$ and hence the α-core of a $G_{\mathcal{M}}$ is nonempty, and that the nonemptiness of the α-core of a $G_{\mathcal{M}}$ is equivalent to that of a $G_{\mathcal{R}}$, which clearly implies the result of Askoura et al. (2013).

Our proofs hinge on an iterated version of Lyapunov's theorem for Young measures to purify partially as well as fully Young measure strategies in an expected payoff function, the derivation of which is a main methodological contribution of this paper.

A finite game with incomplete information, also known as a Bayesian game, consists of finitely many players who know their own types but do not necessarily know the types of the others. The players are assumed to make decisions about their actions according to the Bayesian hypothesis, that is, they form subjective probability distributions on unknown parameters and evaluate their welfare by calculating expected payoffs. In Harsanyi's seminal paper on this subject (1967), such a game was called an "I-game", a game consisting of a finite number of players, each of which is endowed with a type space, action space, payoff function and subjective probability distribution on the others' types conditioned on his/her own type. In other words, in an I-game, the players assess their welfare by computing the conditional expected payoffs with respect to their subjective beliefs about the other players' types given their own types.

In this paper we take for granted the so-called Harsanyi doctrine, asserting the existence of a consistent common prior whereby an I-game can be transformed into a so-called C-game, effectively a normal-form game, in which the players evaluate their ex-ante expected payoffs. Since the aim of this paper is to show that the methods in Scarf (1971) and Kajii (1992) work in our context as well, we do not address the information issues raised by Myerson (2007) and others pertaining to I-games, in which players may have different information right from the outset of the games.

In Askoura et al. (2013), a finite game with incomplete information means an n-player game in which each player i possesses a type space T_i and an action space A_i and chooses a behavioral

E-mail address: noguchi@meijo-u.ac.jp.

http://dx.doi.org/10.1016/j.jmateco.2014.09.006

strategy, a type-dependent mixed action, δ_i, assessing its welfare by the expected payoff.

To be more precise, δ_i is required to be a Young measure, a function $\delta_i : T_i \to \text{Prob}(A_i)$, where T_i is now a measure space, A_i a compact metric space, and $\text{Prob}(A_i)$ the set of probability measures on A_i. Each player i has a type-action dependent payoff function $u_i(t, a)$, where $a = (a_1, \ldots, a_n)$ is an action profile and $t = (t_1, \ldots, t_n)$ a type profile, and also has a common prior belief $\mu(t)$, which governs the manner a chance move selects $t = (t_1, \ldots, t_n)$.

Player i evaluates an expected payoff function in the following form:

$$E_i(\delta) = \int_T \left[\int_A u_i(t, a)\, d(\delta_1 \otimes \cdots \otimes \delta_n)_t(a) \right] d\mu(t),$$

where $A = A_1 \times \cdots \times A_n$, $T = T_1 \times \cdots \times T_n$, and $\delta = (\delta_1, \ldots, \delta_n)$, and where the product Young measure $\delta_1 \otimes \cdots \otimes \delta_n : T \to \text{Prob}(A)$ is defined by $(\delta_1 \otimes \cdots \otimes \delta_n)_{(t_1,\ldots,t_n)} = \delta_{1t_1} \otimes \cdots \otimes \delta_{nt_n}$.

Observe that the above formulation transforms a finite game with incomplete information into one in normal form, and thereby suppresses the incomplete information aspects of the original game.

Section 2 gives an accurate description of the games with incomplete information and investigates the relationship between the expected payoffs of behavioral strategies and those of pure strategies. We rely on repeated use of Lyapunov's theorem for Young measures. Section 3 discusses our main results.

2. Games with incomplete information and α-core

A detailed discussion on Lyapunov's theorem for Young measures can be found in Balder (2000, Theorem 5.10, p. 24) and also in Balder (2008, Theorem 2.1, p. 76). We adopt as much as possible the notations and definitions in Balder (2008) so that the reader can go back and forth readily between this section and Balder's comprehensive treatise on the subject.

2.1. Some facts about Young measures

Let $(T, \mathcal{T}, \mu)$ be a finite and non-atomic measure space and A a metrizable Suslin space. Let $\mathcal{B}(A)$ be the Borel σ-algebra of A and $\text{Prob}(A)$ the set of all probability measures on $(A, \mathcal{B}(A))$. A *Young measure* from $(T, \mathcal{T})$ to $(A, \mathcal{B}(A))$ is defined to be a function $\delta : T \to \text{Prob}(A)$ such that for every $B \in \mathcal{B}(A)$ the function $t \longmapsto \delta_t(B)$ is measurable. Denote by $\mathcal{R}(T; A)$ the set of all such Young measures and by $\mathcal{M}(T; A)$ the set of all measurable functions from $(T, \mathcal{T})$ to $(A, \mathcal{B}(A))$. Note that every $f \in \mathcal{M}(T; A)$ gives rise to a *degenerate Young measure* $\epsilon_f \in \mathcal{R}(T; A)$ defined by $\epsilon_f(t)(B) = 1$ if $f(t) \in B$ and $\epsilon_f(t)(B) = 0$ if $f(t) \notin B$.

As is well-known, there is a close relationship between Young measures from $(T, \mathcal{T})$ to $(A, \mathcal{B}(A))$ and probability measures π on the product space $(T \times A, \mathcal{T} \otimes \mathcal{B}(A))$, where $(T, \mathcal{T})$ is endowed with a probability measure μ: given $\delta \in \mathcal{R}(T; A)$, a product $\mu \otimes \delta$ can be formed by $E \times B \longmapsto \int_E \delta_t(B)\, d\mu(t)$, where $E \in \mathcal{T}$ and $B \in \mathcal{B}(A)$, and conversely, any probability measure π on $(T \times A, \mathcal{T} \otimes \mathcal{B}(A))$ whose marginal on T equals μ can be disintegrated into a product $\mu \otimes \delta$ for some Young measure δ from $(T, \mathcal{T})$ to $(A, \mathcal{B}(A))$. We refer the interested readers to Castaing et al. (2004) for a more detailed discussion in this regard.

A real valued $\mathcal{T} \otimes \mathcal{B}(A)$-measurable function Ψ is called a *Carathéodory integrand* if it is continuous in the second variable a for every $t \in T$ and satisfies $|\Psi(t, a)| \le \phi(t)$ for some μ-integrable real valued function ϕ. Following Balder (1988) et al., the *weak topology* on $\mathcal{R}(T; A)$ is defined to be the coarsest topology for which all functionals of the form $\delta \to \int_{T \times A} \Psi d\mu \otimes \delta$ are continuous, where Ψ is a Carathéodory integrand.

Observe that we may view $\mathcal{M}(T; A)$ as a topological subspace of $\mathcal{R}(T; A)$ in the obvious manner.

In what follows we use the following properties of Young measures (Balder, 2008):

(Y1) Let $\mathbb{R}^n$ be Euclidean n-space and let $H = (H_1, \ldots, H_n) : T \times A \to \mathbb{R}^n$ be a $\mathcal{T} \otimes \mathcal{B}(A)$-measurable map. If $\delta \in \mathcal{R}(T; A)$ satisfies

$$\int_{T \times A} |H|\, d\mu \otimes \delta = \int_T \left[\int_A |H(t, a)|\, d\delta_t(a) \right] d\mu(t) < +\infty,$$

where $|H(t, a)| = \left(\sum_j |H(t, a)|^2 \right)^{\frac{1}{2}}$, then there exists a function $f \in \mathcal{M}(T; A)$ such that

$$\int_{T \times A} H d\mu \otimes \delta = \int_{T \times A} H d\mu \otimes \epsilon_f = \int_T H(t, f(t))\, d\mu(t).$$

(Y2) The subspace $\mathcal{M}(T; A)$ is dense in $\mathcal{R}(T; A)$ (Castaing et al., 2004, Theorem 2.2.3, p. 40).

(Y3) If A is in addition compact, so is $\mathcal{R}(T; A)$ (Castaing et al., 2004, Theorem 4.3.5, p. 92).

2.2. Games with incomplete information

We follow closely Milgrom and Weber (1985) to formulate games with incomplete information. We adopt a minor modification made by Balder (1988) to accommodate general measurable type spaces.

Let $N = \{1, \ldots, n\}$ be a set of n players, each of which has an action space A_i and a measurable type space $(T_i, \mathcal{T}_i)$. For the rest of this paper, A denotes the Cartesian product $\Pi_{i \in N} A_i$ and T the Cartesian product $\Pi_{i \in N}(T_i, \mathcal{T}_i)$, equipped with the product σ-algebra $\mathcal{T} = \otimes_{i \in N} \mathcal{T}_i$. Let μ be a probability measure on the product space $(T, \mathcal{T})$, which governs the random behavior of a type profile $t = (t_1, \ldots, t_n) \in T$. Let μ_i denote the marginal of μ on the factor T_i.

We associate with each player i a payoff function u_i, which depends jointly on a type profile $t = (t_1, \ldots, t_n)$ and an action profile $a = (a_1, \ldots, a_n)$, i.e., $u_i(t, a)$ is a real valued function on $T \times A$.

We call an element $f_i \in \mathcal{M}(T_i; A_i) \subset \mathcal{R}(T_i; A_i)$ a *pure strategy* for player i and an n-tuple $f = (f_1, \ldots, f_n) \in \Pi_{i \in N} \mathcal{M}(T_i; A_i)$ a *pure strategy profile*.

A *coalition* refers to any nonempty subset $S \subset N$. For each coalition S, we write for brevity $\mathcal{M}_S = \Pi_{i \in S} \mathcal{M}(T_i; A_i)$ and $\mathcal{R}_S = \Pi_{i \in S} \mathcal{R}(T_i; A_i)$. $-S$ denotes the complementary coalition of S.

For the rest of this paper, we adhere to the following convention: Let X_i be a set given for player $i \in N$ and let $S \subset N$. We write $X_S = \Pi_{i \in S} X_i$ and for each $x \in X_N$, $x_S \in X_S$ denotes the obvious restriction. For $x^S, y^S \in X_S$, any relation between x^S and y^S must be understood coordinate wise; $x^S \ge y^S$ means $x_i^S \ge y_i^S$ for all $i \in S$. If $S \subset N$ is a coalition then for each $x \in X_N$ and $y^S \in X_S$, $\left(x \mid y^S\right)$ denotes the element $z \in X_N$ such that $z_i = y_i^S$ for all $i \in S$ and $z_i = x_i$ for all $i \in -S$. For each pair $x^S \in X_S$ and $y^{-S} \in X_{-S}$, $\left(x^S, y^{-S}\right)$ denotes the element $z \in X_N$ such that $z_i = x_i^S$ for all $i \in S$ and $z_i = y_i^{-S}$ for all $i \in -S$.

We introduce the following shorthand notations: For each $i = 1, \ldots, n$, let $\mathfrak{X}_i$ be a σ-algebra of subsets of X_i and τ_i a probability measure on $\mathfrak{X}_i$. Let $\otimes_{i=1}^n \mathfrak{X}_i$ be the product of the n σ-algebras and $\otimes_{i=1}^n \tau_i$ the product of the n probability measures. For any $S \subset N$, we write $\mathfrak{X}_S = \otimes_{i \in S} \mathfrak{X}_i$ and $\tau_S = \otimes_{i \in S} \tau_i$.

We state our assumptions as follows:

(A1) For each $i \in N$, A_i is a compact metric space, which is also a convex subset of some linear space.