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Optimal lottery[☆]

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ABSTRACT

This article proposes an equilibrium approach to lottery markets in which a firm designs an optimal lottery to rank-dependent expected utility (RDU) consumers. We show that a finite number of prizes cannot be optimal, unless implausible utility and probability weighting functions are assumed. We then investigate the conditions under which a probability density function can be optimal. With standard RDU preferences, this implies a discrete probability on the ticket price, and a continuous probability on prizes afterwards. Under some preferences consistent with experimental literature, the optimal lottery follows a power-law distribution, with a plausibly extremely high degree of prize skewness.

1. Introduction

The popularity of commercial lotteries offering big prizes with small probabilities reveals a demand for positively skewed lotteries. Skewness preferences arise if gamblers overweight the upper tail of probability distributions. They are commonly modeled by rank-dependent expected utility (Quiggin, 1982), a leading theory of choice under risk, in which agents transform cumulative probabilities.

This explanation provides an intuitive account of lottery demand but falls short as a comprehensive description of lottery markets as the supply side of lotteries is generally not considered. In particular, it remains unclear which exact form should take a lottery (minimal prize, number of tickets, degree of skewness, etc.) compatible with operators maximizing their profit. An equilibrium approach to lottery market is also a first necessary step towards more applied or regulatory issues like tax efficiency and deadweight loss of lottery games, consequences of legal monopolies, existence of scale economies, price and income elasticities of demand, optimal prize structure, to name a few (see Grote and Matheson, 2011 for a recent survey on those issues).

This article aims at filling part of the gap by investigating two related issues: which preference patterns are compatible with profit-maximizing lotteries endowed with multiple prizes and which form take optimal lotteries when consumers are characterized by realistic Rank-Dependent Expected Utility (RDU) preferences? By allowing the lottery operator to freely choose prize values, their probability, and the number of prizes, we show that a finite number of prizes would require both the utility and the weighting function to turn concave then convex or vice-versa each time a new prize is added to the lottery. In a two-payoff lottery, the utility function has a concave–convex–concave shape as in Friedman and Savage (1948) who study the expected utility case. With more than two payoffs, the more realistic case, the number of alternations of the curvature of the two functions becomes implausible.

However, and this is the second part of our paper, RDU preferences naturally fit with continuous probability distributions. In particular, if the utility function is concave, and the weighting function has an inverse-S shape, as the empirical literature suggests, there will be a mass of probability on the worst outcome (paying the ticket price), and the probability distribution will be continuous afterwards. A fundamental characteristic of lottery games is also their very high degree of positive skewness with extremely large jackpots offered with close-to-zero probabilities. We show that prizes over the continuous part of the distribution optimally follow a power-law distribution when realistic functionals for the utility and the weighting functions are chosen. We illustrate our result with a calibration exercise which uses prize data from Euromillions, a Europe-wide lottery game. We document a very high degree of skewness and show its consistency with reasonably calibrated RDU preferences.

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Our approach is in the spirit of Friedman and Savage (1948) who rationalize the demand for lottery tickets in the expected utility (EU) framework. Assuming an increasing marginal utility for a broad range of wealth provides a rationale for two-outcome lotteries, but accounts for very limited patterns of gambling. Markowitz (2010) shows that EU is unable to explain the existence of optimal lotteries with more than two payoffs. We extend his negative result in the more general RDU framework with an arbitrary – yet discrete – number of prizes. Quiggin (1991) also studies the optimal shape of a lottery in RDU and shows the possibility of lotteries with multiple prizes. His argument is however developed in a simplified setup with an exogenous number of equally probable tickets. Once the number of prizes and their probability are made endogenous, we show that a lottery operator has always the incentives to add new prizes between existing ones, provided implausible preference patterns are excluded. Hence the only equilibrium outcome is a continuous prize distribution. Barberis (2012) uses cumulative prospect theory, a variant of RDU, to explain the demand for casino gambling in an intertemporal setup. He does not endogenize the prize structure.

Our results are also related to the burgeoning literature which shows evidence or examine implications of a demand for skew. Garrett and Sobel (1999) and Astebro et al. (2011) show evidence that consumers favor skewness rather than risk in lottery games. Asterbro et al. experimentally explain skewness preferences by small probabilities overweighting (inverse S-shaped weighting function) rather than risk loving (convex utility function). Snowberg and Wolfers (2010) investigate the favorite-longshot bias in horse races betting and reach similar conclusions. Barberis (2013) mentions several articles in which RDU models explain a demand for skewness. In financial markets Barberis and Huang (2008) show that probability weighting may explain why positively skewed securities are overpriced at equilibrium. We contribute to this literature by showing that probability weighting is a better candidate than risk loving to explain multiple-prize lotteries. The optimal degree of skewness is the result of two opposite psychological factors. On the one hand, a concave utility function makes more costly the spread of prize payouts and the inclusion of extreme payoffs. On the other hand a convex weighting function for cumulative probabilities close to one leads consumers to overweight small probabilities associated with extreme prizes, which strengthens the demand for skew. We also link probability weighting with a demand for power-law distributions which have attracted much attention (see e.g. Gabaix, 2011).

Last, our setting is broadly connected to the literature which studies risk sharing between non-expected-utility consumers. Chateauneuf et al. (2000) examine risk sharing arrangement between risk-averse Choquet expected utility agents in special cases. Dana and Carlier (2008) extend the analysis to a broad class of non-expected utility models. Bernard et al. (forthcoming) analyze the optimal insurance contract problem between a risk neutral agent and a RDU agent with an inverse S-shaped probability weighting function. A fundamental difference between those articles and the present one is that consumption risk does not preexist in our environment. To fulfill the demand for risk taking by RDU agents, the lottery operator makes monetary transfers contingent to a "randomization device". In real world, this could be a rotating ball cage or scratch cards.

The presentation is organized as follows. Section 2 studies to what extent RDU preferences may explain the existence of optimal lotteries endowed with a discrete number of prizes. Section 3 reverses the perspective and analyzes the properties of optimal lotteries under realistic RDU preferences. The last section concludes.

2. Optimal discrete lotteries

We analyze in this section which type of preferences is consistent with profit-maximizing firms offering lotteries with a finite number of payoffs (or discrete lotteries for short). The possibility of continuous lotteries is considered in the next section.

2.1. The model

A lottery consists of n payoffs $(x_i)_{i=1,\dots,n}$, and $n+1$ cumulative probabilities $(\pi_i)_{i=0,1,\dots,n}$. Payoffs belong to an interval I (which can be the set of real numbers $\mathbb{R}$, or bounded above or below, with closed or open ends). π_i is the probability that the consumer gets x_i or less (with $\pi_0 = 0$ and $\pi_n = 1$). In a commercial lottery, payoffs are prizes net of the ticket price and the smallest payoff(s) is (are) negative to ensure a positive profit to the firm. The profit of the risk-neutral firm selling the lottery writes:

$$\Pi = -\sum_{i=1}^{n}(\pi_i - \pi_{i-1})x_i.$$

Consumers are RDU decision makers.[1] Both nonlinear weighting of probabilities and nonlinear utility influence risk preferences:

Definition 1. Denote u a strictly increasing and continuously differentiable function on I. Let g be a strictly increasing and continuously differentiable from $[0, 1]$ to itself, satisfying $g(0) = 0$ and $g(1) = 1$. The agent is RDU with utility function u and probability weighting function g if the value U he derives from the lottery writes:

$$U = \sum_{i=1}^{n}(g(\pi_i) - g(\pi_{i-1}))\,u(x_i).$$

Instead of analyzing a monopoly firm maximizing profit under the participation constraint of the consumer, we look at the dual problem of maximizing the player's utility subject to a minimum profit for the firm. This is done for convenience reason, as it is strictly equivalent. In both cases we are looking for a Pareto optimum.[2]

The lottery $\{x_i, \pi_i; i = 1, \dots, n\}$ is optimal if player's utility is maximized under the constraints that the firm obtains at least a profit equal to B, and that π_i and x_i are increasing:

$$\max_n \begin{cases} \max \displaystyle\sum_{i=1}^{n}[g(\pi_i) - g(\pi_{i-1})]u(x_i) \\ \text{s.t. } -\displaystyle\sum_{i=1}^{n}(\pi_i - \pi_{i-1})x_i = B \\ x_{i+1} - x_i \geq 0, \quad i = 1, \dots, n-1 \\ \pi_i - \pi_{i-1} \geq 0, \quad i = 1, \dots, n \\ \pi_0 = 0, \qquad \pi_n = 1. \end{cases}$$

Remark 1. Payoffs and cumulative probabilities must be increasing sequences so as to satisfy RDU preferences and definition of cumulative probabilities. Every time one of the two ordering constraints binds, the number of distinct payoffs included in the lottery is reduced by one. This is obviously the case if $x_{i+1} = x_i$, where the two payoffs have the same probability equal to $\pi_{i+1} - \pi_{i-1}$. If $\pi_i = \pi_{i-1}$ the probability of winning x_i is simply zero. In both cases, removing the ith prize does not change the nature of the lottery.

[1] RDU model is a simple and powerful generalization of the expected utility model. It is able to explain the behavior observed in the Allais paradox, as well as for the observation that many people both purchase lottery tickets and insure against losses. See Machina (1994) and Diecidue and Wakker (2001) for an introduction to RDU theory.

[2] In every allocation problem with non-satiated preferences, when searching for a Pareto equilibrium, it is equivalent to set a minimum utility for the first agent, and maximize the utility of the second agent, or to do the opposite. In particular, it does not matter much how the surplus from trade is split between the firm and the consumer.