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An extension of the Sard–Smale Theorem to convex domains with an empty interior[☆]

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ABSTRACT

A stumbling block in the modeling of competitive markets with commodity and price spaces of infinite dimensions, arises from having positive cones with an empty interior. This issue precludes the use of tools of differential analysis, ranging from the definition of a derivative, to the use of more sophisticated results needed to understand determinacy of equilibria and, more generally, the structure of the equilibrium set. To overcome these issues, this note extends the Preimage Theorem and the Sard–Smale Theorem to maps between convex sets that may have an empty interior.

1. Introduction

Two closely related mathematical results, the Preimage Theorem and Sard's Theorem, are useful tools with many applications in economics, particularly within the theory of general economic equilibrium. The first of these, is stated as follows:

Theorem (*Preimage Theorem*). *If y is a regular value of the map $f : M \to N$ between differentiable manifolds M and N, then $f^{-1}(y)$ is a submanifold of M.*[1]

The Preimage Theorem is usually applied to show results of the following nature:

Consider the excess demand function $Z : \Omega \times S \to X$ of a pure exchange economy, where Ω is the set of parameters (e.g. initial endowments), S is the set of prices and X is the commodity space. Then, if 0 is a regular value of Z, we have that the equilibrium set, $Z^{-1}(0)$, is a manifold.

The set $\Gamma = Z^{-1}(0)$ is called the "equilibrium manifold" and the seminal paper of Balasko (1975) introduced this point of view of general equilibrium theory. The second result in this spirit is Sard's Theorem that states that almost all values of a function are regular. Formally:

Theorem (*Sard's Theorem, 1942*). *Let U be an open set of $\mathbb{R}^p$ and $f : U \to \mathbb{R}^q$ be a C^k map where $k > \max(p - q, 0)$. Then, the set of critical values in $\mathbb{R}^q$ has measure zero.*

Sard's Theorem also has many applications, usually to show results similar to this:

Consider the equilibrium manifold $\Gamma \subset \Omega \times S$ and the projection map $\pi : \Omega \times S \to \Omega$, restricted to Γ, given by $\pi(\omega, p) = \omega$. Then, the regular values are almost all of Ω. In other words, almost all equilibria are determinate.

Sard's Theorem turned out indeed to be the appropriate tool to study determinacy of equilibria since Debreu's (1970) seminal paper.

In spite of these general results and vast applications, many models of competitive markets have an infinite number of commodities which naturally lead to consumption and price spaces of infinite dimensions.[2] At a first glance, it would seem appropriate

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1 Recall that for a map $f : M \to N$, a point x in M at which the derivative of f has rank less than $\dim(N)$ is called a *critical point* and its image a *critical value* of f. Other points y in N, that is, such that f has rank $\dim(N)$ at all points in $f^{-1}(y)$, are called *regular values* of f.

2 The reader may wish to consult (Zeidler, 1993) for an introduction to nonlinear functional analysis.

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to use Smale's generalization of Preimage and Sard's Theorems to infinite dimensions, as follows:

Theorem (*Smale's Theorem, 1965*). *If $f : M \to V$ is a C^s Fredholm map between differentiable manifolds locally like Banach spaces with $s > \max(\text{index} f, 0)$, then*

1. *For almost all $y \in V$, $f^{-1}(y)$ is a submanifold of M;*
2. *The regular values of f are almost all of V.*

The statement and proof of Smale's Theorem is local, and requires for M and V to have a nonempty interior. However, there are many instances in economic modeling that require a domain with an empty interior in which case Smale's Theorem cannot be applied. For example, many models of competitive markets use a consumption space in the positive cone of an ℓ_p or L_p space, for $1 \le p \le \infty$.[3] Unfortunately, the only spaces among L_p and ℓ_p spaces whose positive cone has a nonempty interior are L_∞ and ℓ_∞. To complicate things, prices are elements of the positive cone of the dual space of the commodity space.[4] Recall that the dual space of ℓ_p (L_p, respectively), $1 \le p < \infty$, is the space ℓ_q (L_q, respectively) where $1/p + 1/q = 1$. This implies that even if the commodity space had a positive cone with a nonempty interior, the positive cone of the dual space – that is, the price space – will have an empty interior, and vice versa. The dual spaces of L_∞ and ℓ_∞ are subtler, but this problem still holds.[5]

The purpose of this note is precisely to extend Sard's and Smale's Theorems to maps between subsets of Banach manifolds which may have an empty interior. To this end, we will prove in the next sections the following result.

Theorem (*Main Theorem*). *Let M^* and N^* be star-Banach manifolds, and $f : M^* \to N^*$ be a C^{*r} star-Fredholm map, with $r > \max(\text{index} f, 0)$. Suppose that M^* and N^* are connected and have a countable basis. Furthermore, suppose that f is locally proper and that it has at least one regular value. Then, the regular values of f are almost all of N^*.*

2. Literature overview

There are several examples of economies with an infinite number of commodities. We mention a few below, but it should not be understood as an exhaustive list.

A pioneering work in the modeling of financial markets in an Arrow–Debreu setting with uncertainty is due to Duffie and Huang (1985). In this paper, they consider a wide class of information structures, including those generated by continuous-time state-variable stochastic processes. The natural consumption space in this setting is the positive cone of the space L_2 since it restricts the analysis to consumption claims with finite variance. Notice that an important characteristic of this framework is that prices are also elements of the positive cone of L_2 (since the dual space of L_2 is itself). This means that both the consumption and price sets will have an empty interior. Duffie and Huang (1985) do not study the structure of the equilibrium set of its model, but this model is implicitly included in Kehoe et al. (1989) although they need to allow for negative prices and negative consumption. Our paper overcomes this issue in Section 5 by studying the structure of the equilibrium set without the assumption of negative prices or consumption.

Another example is the infinite horizon model, represented through consumption streams in the set ℓ_∞, that is, at every moment of time the consumption of each agent is bounded. Kehoe and Levine (1985) show that equilibria are locally unique but do not study the structure of the equilibrium set. Balasko (1997a,b,c) improves these results by showing that the equilibrium set is a manifold, but requires utilities to be separable and restricts the analysis to "truncated economies". These papers use the Negishi method to substitute the study of price equilibria to the study of welfare weights. This approach requires welfare theorems to hold while our paper directly studies equilibrium prices.

Bewley (1972) uses the space L_∞ to model infinite variations in any of the characteristics describing commodities. These characteristics could be physical properties, location, the time of delivery, or the state of the world (in the probabilistic sense) at the time of delivery. Bewley imposes the Mackey topology to consumer's preferences in order to ensure that prices are elements of L_1. While he shows existence of equilibria, the structure of the equilibrium set is unknown as far as the authors are aware.

Similarly, Chichilnisky and Zhou (1998) study an "approximative" model of competitive markets with infinite dimensional commodity spaces. Since continuous functions are dense in all L_p spaces, that is all functions in L_p can be approximated arbitrarily close by a continuous function, they let the consumption space to be $C(K)$ for some compact set K. By supposing separable utilities, they show that prices are also elements of $C(K)$. In this setting Covarrubias (2010, 2011) shows that the price equilibrium set is a manifold and provides a topological characterization. Similarly, Accinelli (2013) shows that the social equilibrium set is a manifold and also provides a topological characterization. This second paper incorporates stronger results since it does not require consumption or price spaces to be restricted to $C(K)$.

At this point it is important to mention that Shannon (1999) and Shannon and Zame (2002) have studied the question of determinacy of equilibria in infinite dimensions in much detail, and their results are applicable to all models previously mentioned. Indeed, they show that if utilities satisfy "quadratic concavity", equilibria are then generically determinate. Our results on determinacy are strictly speaking not comparable since we do not take utilities as primitives and hence we do not require them to satisfy quadratic concavity. Even so, these two papers do not explore the equilibrium set that we are able to do by proving a directional Preimage Theorem.

3. Analytical preliminaries

3.1. Star topology

Let $(B, \ge)$ be a Banach lattice space, and let B_+ denote the positive cone of B, i.e.,

$$B_+ = \{x \in B : x \ge 0\},$$

which may have an empty interior. Notice that B_+ is a convex subset of B. The results of this paper can be generalized to any convex subset, not just the positive cone, but we restrict the analysis to this set because of an interest in economic applications.

[3] Recall the following definitions. Let p be a real number $1 \le p < \infty$. The space ℓ_p consists of all sequences of scalars $\{x_1, x_2, \dots\}$ for which $\sum_{i=1}^{\infty} |x_i|^p < \infty$. The norm of an element $x = \{x_i\}$ in ℓ_p is defined as $\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$. The space ℓ_∞ consists of bounded sequences. The norm of an element $x = \{x_i\}$ in ℓ_∞ is defined as $\|x\|_\infty = \sup_i |x_i|$. The L_p spaces are defined analogously. For $p \ge 1$, the space $L_p[a, b]$ is the set of all classes of real-valued, Lebesgue-measurable functions x defined on the interval $[a, b]$ for which $\int_a^b |x(t)|^p < \infty$. The norm on this space is defined as $\|x\|_p = \left(\int_a^b |x(t)|^p\right)^{1/p}$. The space $L_\infty[a, b]$ consists of all classes of real-valued, Lebesgue-measurable functions on $[a, b]$ which are bounded, except possibly on a set of measure zero. The norm is defined by $\|x\|_\infty = \text{ess sup}|x(t)|$.

[4] If X is a normed linear vector space, the space of all bounded linear functionals on X is called the *normed dual* of X and is denoted by X^*. The norm of an element $f \in X^*$ is $\|f\| = \sup_{\|x\|=1} \|f(x)\|$.

[5] The dual space of L_∞ can be identified with bounded signed finitely additive measures that are absolutely continuous with respect to the Lebesgue measure.