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The instability of some non-full-support steady states in a random matching model of money

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ABSTRACT

Zhu (2003) shows existence of full-support monetary steady states with strictly concave value functions in a random matching model with individual money holdings in $\{0, 1, 2, \ldots, B\}$ for a general B. He also shows that corresponding to each such steady state is an l-replica steady state for each $l \in \mathbb{N}$: money is traded in bundles of l units, the support is $\{0, l, 2l, \ldots, lB\}$, and the value function is a step-function with jumps at points of the support. We show that such l-replicas are unstable if the underlying full-support steady state is a pure strategy steady state and if the support of the initial distribution is not $\{0, l, 2l, \ldots, lB\}$.

1. Introduction

Shi (1995) and Trejos and Wright (1995) study a matching model of fiat money with individual money holdings in the set $\{0, 1\}$. Such holdings are special because the distribution of holdings is determined by the stock of money; that is, it is unaffected by the trades that are made. That property disappears for any richer set of individual money holdings. For buyer take-it-or-leave-it offers, Zhu (2003) studies the model with individual money holdings in the set $\{0, \Delta, 2\Delta, \ldots, B\Delta\}$, for arbitrary B. He provides sufficient conditions for the existence of a steady state with a full-support money-holding distribution and a strictly increasing and strictly concave value function, a steady state that we call a *Zhu steady state*.

In Zhu's model, there are three exogenous nominal quantities: $(\Delta, B\Delta, m)$, where m is the per capita stock of money. If, for some positive integer $l \geq 2$, we compare that economy to an otherwise identical economy with nominal quantities $(l\Delta, lB\Delta, lm)$, then we have neutrality. But what if we compare $(\Delta, B\Delta, m)$ to $(\Delta, lB\Delta, lm)$? Zhu shows that any steady state for $(\Delta, B\Delta, m)$ is also a steady state for $(\Delta, lB\Delta, lm)$, the one in which all owned/traded quantities of money are multiplied by l and the value function is a step function with steps at and only at integer multiples of an l-bundle. We call such a non-full-support steady state an *l-replica*. In an l-replica with $l = 10$, for example, ten \$1's are treated only as a bundle; that is, as a \$10. Such a steady state implies a lower real balance of money and, almost certainly, lower welfare than in the full-support steady state of $(\Delta, lB\Delta, lm)$.

The presence of l-replicas complicates the use of the model for policy analysis unless there are reasons to ignore them. Wallace and Zhu (2004) show that such replicas are not robust to the introduction of a small utility of holding money. Here we show something even stronger; if l-replicas are constructed from a Zhu steady state that is supported by pure strategies, then they are not stable. Specifically, if the initial distribution has support different from $\{0, l\Delta, 2l\Delta, \ldots, Bl\Delta\}$, then there is no equilibrium that converges to such l-replicas.

Zhu steady states can either be pure-strategy steady states or mixed strategy steady states and, as just noted, our instability result applies only to l-replicas that are constructed from pure-strategy steady states.[1] It remains an open question whether the result extends to l-replicas that are constructed from mixed-strategy steady states.

One reason to study the Zhu (2003) model is that it has policy implications that differ from those of the model with money holdings in $\{0, 1\}$ and from models with degenerate distributions

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[1] See Huang and Igarashi (2012) for a demonstration that both kinds are generic. For any B, a pure-strategy Zhu steady state exists for a sufficiently high β. See Camera and Corbae (1999).

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of money holdings. In particular, as discussed in Wallace (2014) and shown by Molico (2006) and Deviatov (2006), moderate inflation improves welfare through re-distributional effects in versions of the Zhu model. That cannot happen in models with money holdings in $\{0, 1\}$ or in models with degenerate distributions of money holdings.

2. Model

The model is that in Zhu (2003). Time is discrete, dated as $t \geq 0$. There is a non-atomic unit measure of infinitely-lived agents. There is a consumption good that is perfectly divisible and perishable. Each agent maximizes the expected discounted utility with discount factor $\beta \in (0, 1)$. Utility in a period is $u(c) - q$, where $c \in \mathbb{R}_+$ is the amount of good consumed and $q \in \mathbb{R}_+$ is the amount of good produced. $u : \mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing and strictly concave, and satisfies $u(0) = 0$ and $u'(\infty) = 0$. In addition, $u'(0)$ is sufficiently large but finite.

There is a fixed stock of intrinsically useless money that is indivisible and perfectly durable. Because of the neutrality we discussed in the previous section, the size of the smallest unit of money is normalized to one. Two other exogenous quantities are (B, m), where B is the maximum units that an agent can hold, and $m \in (0, B)$ is the fixed per capita stock of money. We denote the set of possible individual money holdings by $\mathbb{B} = \{0, 1, \ldots, B\}$. The state of the economy at each date is a distribution over $\mathbb{B}$, which for each $k \in \mathbb{B}$ gives the fraction of agents who have k units of money.

In each period, agents are randomly matched in pairs. With probability $1/N$, where $N \geq 2$, an agent is a consumer (producer) and the partner is a producer (consumer). Such meetings are called single-coincidence meetings. With probability $1 - 2/N$, the match is a no-coincidence meeting.[2] In meetings, agents' money holdings are observable, but any other information about an agent's trading history is private.

In a single-coincidence meeting between a consumer with i units of money and a producer with j units of money, an (i, j)-meeting, the consumer makes a take-it-or-leave-it offer consisting of the amount to be produced, q, and the amount of money to be paid, p. The offer must be feasible, $0 \leq p \leq \min\{i, B - j\}$, and must satisfy the producer's participation constraint, $-q + \beta w_{j+p}^{t+1} \geq \beta w_j^{t+1}$, where w_k^t is the expected discounted value of holding $k \in \mathbb{B}$ units of money, prior to date-t matching. Because the optimal offer leaves no positive gain to the producer, the consumer's problem reduces to choosing p in the feasible set of offers of money

$$p^t(i, j, w^{t+1}) = \underset{0 \leq p \leq \min\{i, B-j\}}{\operatorname{argmax}} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \quad (1)$$

Because $p^t(i, j, w^{t+1})$ is discrete and may be multi-valued, randomization over the elements of $p^t(i, j, w^{t+1})$ is allowed. Let $\lambda^t(p; i, j)$ be the probability that consumers with i (pre-trade) in meetings with producers with j offer p at date t. It has support in $p^t(i, j, w^{t+1})$ in equilibrium, so that

$$\sum_{p \in p^t(i,j,w^{t+1})} \lambda^t(p; i, j) = 1. \quad (2)$$

Let π_k^t denote the fraction of agents holding k units of money prior to date-t matching. The law of motion is

$$\begin{aligned} \pi_k^{t+1} = \pi_k^t &+ \frac{1}{N} \sum_{\{i,j|i>k\}} \pi_i^t \pi_j^t \lambda^t(i-k; i, j) \\ &+ \frac{1}{N} \sum_{\{i,j|j<k\}} \pi_i^t \pi_j^t \lambda^t(k-j; i, j) \\ &- \frac{1}{N} \sum_j \pi_k^t \pi_j^t \sum_{p>0} \lambda^t(p; k, j) \\ &- \frac{1}{N} \sum_i \pi_i^t \pi_k^t \sum_{p>0} \lambda^t(p; i, k). \end{aligned} \quad (3)$$

The Bellman equation is

$$\begin{aligned} w_i^t = \frac{N-1}{N} \beta w_i^{t+1} &+ \frac{1}{N} \sum_{j=0}^{B} \pi_j^t \sum_p \lambda^t(p; i, j) \\ &\times \left\{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\right\}. \end{aligned} \quad (4)$$

The first term on the r.h.s. corresponds to entering a no-coincidence meeting or becoming a producer who is indifferent between accepting and rejecting the offer. Free disposal of money is permitted which implies that the value function must be nondecreasing in every period:

$$w_i^t \geq w_{i-1}^t, \quad \text{for } i = 1, \ldots, B, \text{ and } w_0^t = 0. \quad (5)$$

Given (5), we focus on equilibria in which agents do not dispose of money.

Definition 1. Given π^0, an *equilibrium* is a sequence $\{(\lambda^t, \pi^t, w^t)\}_{t=0}^{\infty}$ that satisfies (1)–(5). A *monetary steady state* is (λ, π, w) with $w \neq 0$ such that $(\lambda^t, \pi^t, w^t) = (\lambda, \pi, w)$ for all t is an equilibrium. Pure-strategy steady states are those for which (1) has a unique solution for all meetings. Other steady states are called mixed-strategy steady states.[3] A *Zhu steady state* is a steady state for which π has a full support and w is strictly increasing and strictly concave.[4]

Our definition of steady states differs from that in Zhu only in that we explicitly include λ, a description of trades. In a non-full support steady state, some meetings occur with zero probability. Our definition requires a definition of trades for all meetings, including those which occur with zero probability. That is because such meetings will in general occur with positive probability near a steady state.

Next, we formally define l-replicas for economies in which the smallest unit of money is one.

Definition 2. Let $s = (\hat{\lambda}, \hat{\pi}, \hat{w})$ be a Zhu steady state of economy (B, m). For integer $l \geq 2$, an l-replica of s, denoted by $s(l) = (\lambda^*, \pi^*, w^*)$, is a steady state of economy (lB, lm) that satisfies

$$\pi_{il}^* = \hat{\pi}_i, \quad \text{and} \quad \pi_{il+i'}^* = 0, \quad \forall i' \in \mathbb{L}, \quad (6)$$

$$w_{il}^* = \hat{w}_i, \quad \text{and} \quad w_{il+i'}^* = w_{il}^*, \quad \forall i' \in \mathbb{L}, \quad (7)$$

where $\mathbb{L} \equiv \{1, \ldots, l-1\}$.

When we discuss an l-replica or convergence to an l-replica, if an agent holds $il + i'$ units of money for some $i \in \{0, 1, \ldots, B-1\}$ and $i' \in \mathbb{L}$, we say that the agent has i "*bundles*" and i' units of "*change*."[5]

The following is our main result.

[2] One foundation is that there are N types of agents and N types of consumption goods, that type-n agents can produce type-n goods only and consume type-$(n+1)$ goods only, and that the money is symmetrically distributed across the types.

[3] Note that in this definition, the 'boundary' situations in which (1) has more than one solution but in which the randomization is degenerate are included in mixed-strategy equilibria. Such situations are non-generic.

[4] Zhu (2003) gives sufficient conditions for the existence of such steady states.

[5] In what follows, non-prime letters (i, j, etc.) indicate numbers of bundles, and letters with primes ($i', j' \in \mathbb{L}$, etc.) indicate units of change.