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Semi-implicit representations of surfaces in $\mathbb{P}^3$, resultants and applications[☆]

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Abstract

In this paper we introduce an intermediate representation of surfaces that we call semi-implicit. We give a general definition in the language of projective complex algebraic geometry, and we begin its systematic study with an effective view-point. Our last section will apply this representation to investigate the intersection of two bi-cubic surfaces; these surfaces are widely used in Computer Aided Geometric Design.

Keywords: Algebraic family of curves; Resultants; Determinantal varieties

1. Introduction

Parametric and implicit representations of surfaces in $\mathbb{R}^3$ offer complementary advantages for the applications in engineering, specially in Computer Aided Geometric Design (CAGD for short). The parametric representation presents the surface as the image of a rational map from $\mathbb{R}^2$ to $\mathbb{R}^3$; this allows fast generation of points on the surface and flexibility for designing. The implicit representation defines an algebraic constraint used to

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determine if a point belongs to a surface S and provides indications to locate it in case it is outside S; implicit representation is also useful for surface blending. Intersection of two surface patches can be done accurately if one patch is given by a parametric representation and the other by an implicit representation. Unfortunately, conversion from one representation to the other is not always possible and when possible it is, in general, difficult and costly.

In this paper we introduce an intermediate representation of surfaces that we call *semi-implicit*. We give a general definition in the language of projective complex algebraic geometry, and we begin its systematic study with an effective view-point. Our last section will apply this representation to investigate the intersection curve (which is of degree 324) of two bi-cubic surfaces which are widely used in CAGD.

Our starting observation is the following: a tensor-product parametric surface can be viewed as the projection S in $\mathbb{P}^3$ of the graph $\mathcal{G}$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$ of a rational map, whereas our object of study will be the intermediate projection $\mathcal{Z}$ in $\mathbb{P}^1 \times \mathbb{P}^3$. This is a surface of codimension 2 in $\mathbb{P}^1 \times \mathbb{P}^3$ fibred over $\mathbb{P}^1$.

We state formal definitions and expected properties. Although the geometry and representation of $\mathcal{Z}$ can be complicated, we single out the special case of surfaces spanned by a family of determinantal curves, for which we derive useful formulae and algorithms. For that purpose we will use an adapted generalized resultant which provides a compact determinantal representation for the corresponding implicit equation. The fact that we obtain this polynomial equation via a resultant is a guaranty of a good numerical stability of the output and allows reliable approximate computations.

The paper is organized as follows. Section 2 describes our setting, defines formally semi-implicit representations of a reduced surface in $\mathbb{P}^3$, states two basic problems and illustrates them. Section 3 describes the needed algebraic tools, including a generalized resultant, and show how to use them to manipulate these semi-implicit representations. Section 4 is devoted to the application of our results to the study of the intersection of two bi-cubic surface patches: we view such surfaces as families of determinantal curves.

We will always work over the algebraically closed field $\mathbb{C}$, unless specified in the text.

2. Semi-implicit representation of surfaces in $\mathbb{P}^3$

An implicit representation of a surface S in $\mathbb{P}^3$ consists in viewing it as a closed subvariety of $\mathbb{P}^3$, i.e. describing it as the zero locus of a collection of homogeneous polynomials in $\mathbb{C}[x, y, z, w]$. In this section we represent surfaces in $\mathbb{P}^3$ in a different way, as *parametrized* families of *implicitly* represented space curves. We call such a representation a *semi-implicit* representation. It basically consists in viewing a surface $S \subset \mathbb{P}^3$ as the projection on the second factor of a certain closed subvariety $\mathcal{Z}$ of $\mathbb{P}^1 \times \mathbb{P}^3$. We restrict our study to the case of reduced pure dimensional surfaces, i.e. not necessarily irreducible but each component is a surface occurring with multiplicity 1.

Before stating a formal definition, let us recall few facts about space curves. Opposed to a parametrized representation which only exists for a rational curve, an implicit representation may represent any space curve. The more common way to describe implicitly a space curve is to give a (minimal) set of generators, say homogeneous