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Spanning with indexes

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ABSTRACT

This paper presents several approximation theorems of a general contingent claim in terms of index options. We demonstrate that any contingent claim on the primitive securities in an infinite state economy can be approximated arbitrarily close by a portfolio of index options. In addition, these index options are associated with the same payout function, which belongs to a large and explicit class of one-variable measurable functions. I also characterize the layer structure of a general contingent claim.

1. Introduction

In a seminal paper concerning the complexity of securities markets, Ross (1976) showed that plain call and put options written on an index, that is, a portfolio of primitive securities, span the completion of the set of primitive securities in a finite state economy. Ross (1976) also proved that a complete market can be achieved by trading plain options alone on this portfolio if the relevant set of contingencies is represented by the events characterized by the payoffs of the primitive securities. Therefore, when the market information set is entirely generated by the primitive securities, plain options on an index can complete a static securities market in the same way as adding Arrow–Debreu securities in an incomplete economy. Moreover, the underlying index can be chosen randomly as the set of indexes with such a spanning power has a full Lebesgue measure in an appropriate space of portfolios.[1]

Given its apparent importance, there has been considerable interests in the literature to study the spanning problem in a general setting to encompass asset pricing models in finance. For instance, Galvani (2009) and Galvani and Troitsky (2010) proved that with a Polish state space and an exogenous information set given by the Borel σ-algebra, plain call options on any bounded and resolving contingent claim can complete L_p-spaces, with $1 \leqslant p \leqslant \infty$, under mild conditions.[2] Nachman (1987) showed the existence of one contingent claim and termed it as an efficient fund, which spans the completion of the set of primitive securities if the measurable algebra of the securities market is separable. In addition, under a separable property on the asset space, this efficient fund can be chosen as a portfolio of primitive securities. Regarding the complexity of contingent claim space, Green and Jarrow (1987) showed that all contingent claims written on primitive securities have a finite layer structure. Nachman (1988) further demonstrated that instead of options written on an index, as formulated in Ross (1976), options on portfolios of call options written on individual primitive securities can span the completion of the set of primitive securities. Duan et al. (1992) showed that the index options can approximate any contingent claim on the systematic risk in a linear factor pricing model. See also Brown and Ross (1991) and Galvani (2007).

In this paper, we demonstrate that Ross's original spanning results virtually hold in an approximate sense in a general static securities market, without imposing any technical topological conditions on the asset spaces and the state space. Specifically, we show that any contingent claim on the primitive securities can be approximated arbitrarily close by a portfolio of index options. Furthermore, the payout functions of these index options can be chosen as one explicit, single-argument function, in a fairly large class of measurable functions. (Some examples are displayed in Table 1.) Hence, plain index options span the completion of the set of primitive securities; these index options complete the securities market if the relevant sets of contingencies are represented by the events characterized by the payoffs of the primitive securities. In contrast to plain call options on a resolving contingent claim such as in Ross (1976), Galvani (2009) and Galvani and Troitsky (2010), or on an efficient fund as in Nachman (1987), our approximation results rely on multiple underlying indexes. Each underlying index

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[1] See Arditti and John (1980) and John (1981, 1984). Baptista (2003) examined the spanning problem for American-type contingent claim in a finite state space.

[2] A one-to-one claim is called a resolving contingent claim. Clearly, there is a resolving claim in a finite state space if and only if there exists an injective index.

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Table 1
Index options in approximation theorems.

$\sigma(t)$	Index option	*Approximation theorem*
$\frac{1}{1+e^{-t}}$	Sigmoidal index option	Theorem 2.1
$1_{\{t\geqslant K\}}$	Digital index option	Theorem 2.1
$\max\{x-K,0\}$	Call index option	Theorem 2.7
$\max\{K-x,0\}$	Put index option	Theorem 2.7
$\max\{x-K,0\}-\max\{x-L,0\}$	Call spread index option	Theorems 2.1, 2.7
$\max\{K-x,0\}-\max\{L-x,0\}$	Put spread index option	Theorems 2.1, 2.7
$\max\{x-K,0\}+\max\{L-x,0\}$	Strangle on index	Theorem 2.7
$\max\{x-K_1,0\}-\max\{x-L_1,0\}$ $-\max\{x-K_2,0\}+\max\{x-L_2,0\}$	Condor on index	Theorems 2.4, 2.7
$\cos(t)$	Trigonometric index option	Theorems 2.4, 2.7
$\sin(t)$	Trigonometric index option	Theorems 2.4, 2.7
$\tan^{-1}(\gamma t)$	Trigonometric index option	Theorems 2.4, 2.7
$\cosh^{-1}(t)$	Trigonometric index option	Theorems 2.4, 2.7
e^t	Exponential index option	Theorem 2.5
t	Product index option $\prod(m\cdot x+\theta)$	Theorem 2.5
$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t}e^{-u^2/2}du$	Distribution index option	Theorem 2.1

This table demonstrates some examples of the payoff function $\sigma(t)$ and the corresponding index options. We also report the relevant approximation theorem. In this table, a *sigmoidal index option* is an option with payoff function $\sigma(t)$ satisfying condition (1) in Theorem 2.1. By a *distribution index option* we mean an option with a payoff function $\Phi(m\cdot x+t)$, where $\Phi(t)$ is a cumulative probability distribution of a random variable. In these index options, Theorems 2.5–2.7 ensure that the index space can be reduced by its bounded weights or the integral combinations. We also note that the strike parameter K, L in these examples can be any specific positive number, so we do not need more than one ordinary call or put index option.

has a remarkably simple structure and can be likened to a portfolio of the primitive securities, plus a risk-free bond. The same cannot be said, in general, for resolving assets of efficient funds. In fact, as a resolving contingent claim is built on the exogenous information set, it is somewhat mysterious in regard to its construction.[3]

To demonstrate our spanning results we consider a spanning problem which is different from the classical spanning problem initiated by Ross (1976).[4] The classical spanning problem is to investigate whether plain call and put options of an index span all contingent claims or not. Previous studies show that these plain options have the potential to complete the market and any risk profile can be achieved through a hedging portfolio of plain options. On the contrary, we fix a payout function of index options, and investigate whether these index options written on various indexes can span all contingent claims or not. We propose a new class of derivatives, options written on exchanged traded funds, to complete the market. Therefore, our theoretical results can be regarded as evidence favoring the expansion of the exotic option market.

Our approach to the spanning problem is inspired by the *universal approximation theorem* in the mathematical theory of neural networks, in particular, the works of Cybenko (1989), Funahashi (1989), Hornik et al. (1989), Stinchcombe and White (1990), and Hornik (1991).

The paper proceeds as follows. In Section 2 we present our approximation theorems. All proofs are given in Section 3. In Section 4 we offer our conclusions.

2. Approximate spanning index options

Let $(\Omega,\mathcal{F},P)$ denote a probability space that represents the state of uncertainty. An asset or security is modeled by an $\mathcal{F}$-measurable real-valued function. We study the set of contingent claims as $L^p(\Omega,\mathcal{F},P)$ for a $1\leqslant p\leqslant\infty$ (in Theorems 2.1–2.4), and $L(\Omega,\mathcal{F},P)$ (in Theorems 2.5–2.7).[5] There are a finite number of primitive securities, $x_1,\ldots,x_n$, in the securities market, where each $x_i\in L(\Omega,\mathcal{F},P)$. We study under what conditions any contingent claim in $L^p(\Omega,\mathcal{F},P)$ or $L(\Omega,\mathcal{F},P)$ can be approximated arbitrarily well by index options, say, the limit under the L^p-topology or the almost surely limit of a sequence of index options with form $\sum_{j=1}^{N}\alpha_j\sigma\left(m_j\cdot x+\theta_j\right)$, $m_j\in\mathbb{R}^n$, $\alpha_j,\theta_j\in\mathbb{R}$. Here $m\cdot x+\theta=m_1x_1+\cdots+m_nx_n+\theta$, where $m=(m_1,\ldots,m_n)$, represents an *index* with m_i units on the security x_i and a cash position worth θ dollars.[6] $\sigma(\cdot)$ is a one-variable $\mathbb{R}$-valued function over $\mathbb{R}$, and $\sigma(\cdot)$ denotes the payout function of all index options on index $m\cdot x+\theta$. Many examples of $\sigma(\cdot)$ will be given in Examples 2.1–2.4 and Table 1.

Specifically, let

$$H=\operatorname{span}\left\{\sigma(m\cdot x+\theta),m\in\mathbb{R}^n,\theta\in\mathbb{R}\right\}$$

be a linear spanning space[7] of index options $\sigma(m\cdot x+\theta)$. We examine the spanning power of H and verify $L^p(\Omega,\mathcal{F},P)=\bar{H}$ under a suitable topology in the contingent claim space. Let $\mathcal{F}\{x_1,\ldots,x_n\}$ be a σ-subalgebra of $\mathcal{F}$ generated by the primitive securities, $x_1,\ldots,x_n$.

Theorem 2.1. *Assume that $\sigma:\mathbb{R}\to\mathbb{R}$ is a bounded measurable function satisfying*

$$\lim_{t\to\infty}\sigma(t)=1,\qquad \lim_{t\to-\infty}\sigma(t)=0. \tag{1}$$

1. *For any $1\leqslant p<\infty$, $L^p(\Omega,\mathcal{F}\{x_1,\ldots,x_n\},P)=\bar{H}$, the closure of H under the L^p-norm topology.*

[3] It is equally obscure about the structure of an efficient fund presented in Nachman (1987) and Duan et al. (1992).

[4] See also Nachman (1987, 1988), Green and Jarrow (1987), Galvani (2009), and Galvani and Troitsky (2010).

[5] Brown and Ross (1991), Green and Jarrow (1987), and Ross (1976) considered the spaces of contingent claims in other contexts such as the set of all $\mathcal{F}$-measurable functions, a set of all continuous functions, or a set of all real-valued functions. Following Kreps (1981) and Harrison and Kreps (1979), we consider all assets with a finite pth moment in Theorems 2.1–2.4. We examine the space of measurable functions, $L(\Omega,\mathcal{F},P)$, in Theorem 2.5 through Theorem 2.7 concerning on general contingent claim with respect to almost surely convergence.

[6] θ should be viewed as $\theta 1_\Omega$, where 1_Ω is defined by $1_\Omega(\omega)=1$ for each ω in Ω. 1_Ω is interpreted as the payoff of the risk-free asset or as the payoff of the numeraire in a securities market.

[7] Following Galvani (2009) and Galvani and Troitsky (2010), the element of H contains finitely many components.