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Stability and determinacy conditions for mixed-type functional differential equations[☆]

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ABSTRACT

This paper analyzes the solution of linear mixed-type functional differential equations with either predetermined or non-predetermined variables. Conditions characterizing the existence and uniqueness of a solution are given and related to the local stability and determinacy properties of the steady state. In particular, it is shown that the relationship between the uniqueness of the solution and the stability of the steady-state is more subtle than the one that holds for ordinary differential equations, and gives rise to new dynamic configurations.

1. Introduction

Mixed-type functional differential equations (MFDEs) allow us to describe the dynamics of a variable whose time derivative depends on its past and future values. A great number of dynamic economic problems in continuous time could be written with an MFDE; however, some simplifying assumptions are commonly used in order to reduce the problem to a system of ordinary differential equations (ODEs). As an illustration, the unrestricted form of dynamics of an overlapping generations model[1] results in an MFDE except in the case where exponential forms are retained for the survival, discount, and endowment functions (Blanchard, 1985). Similarly, models that consider lagged price contracts (Whelan, 2007) or vintage capital[2] generally have dynamics characterized by an MFDE. The purpose of this article is to put forward conditions for the uniqueness of the solution and the asymptotic stability of such MFDEs. These conditions are the equivalent for MFDEs of the Blanchard and Kahn conditions that apply to finite-dimensional systems (Blanchard and Kahn, 1980; Buiter, 1984).

The Blanchard and Kahn conditions are based upon the set of initial conditions of the system and a spectral decomposition of the characteristic equation. More precisely, by comparing the dimensions of the space of predetermined variables with those of the stable eigenspace (or equivalently, by comparing the dimension of the space of non-predetermined variables and those of the unstable eigenspace) they characterize the local uniqueness and stability in the neighborhood of a steady state. On the other hand, for functional differential equations, some of the spaces are infinite-dimensional and the Blanchard and Kahn conditions do not apply. This is the case for the stable eigenspace for delay differential equations (DDEs). In the standard case where the variables are predetermined and continuous, there is at most one solution to this type of equation (Diekmann et al., 1995). Otherwise, multiple solutions may arise if the dimension of the space of predetermined variables is greater than that of the unstable eigenspace (d'Albis et al., 2012, 2014). On the contrary, advance differential equations (ADEs) are

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1 Demichelis (2002), Boucekkine et al. (2002), d'Albis and Augeraud-Véron (2007) and Edmond (2008).

2 Benhabib and Rustichini (1991) and Boucekkine et al. (2005).

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characterized by an unstable eigenspace of infinite dimension and it is necessary to compare the dimension of the stable eigenspace with that of the predetermined variables.

The difficulty with MFDEs, which contain both delays and advances, is that both the stable manifold and the unstable manifold are infinite-dimensional. In order to establish our results, we use and extend the results of Mallet-Paret and Verduyn Lunel (in press). This approach consists of analyzing a factorization of the characteristic equation of the MFDE in question, written as the product of two characteristic equations associated with a DDE and an ADE, respectively. Existence and uniqueness of solutions to either differential equation depend on the number of misplaced roots of the respective characteristic function. Taking the difference between these numbers, we are able to provide conditions for existence and uniqueness of solutions to the MFDE, and to characterize the stability properties and degree of indeterminacy in the neighborhood of a steady state. We then extend these results to certain algebraic equations of mixed type.

The advantage of the technique we propose is that it is simple enough to implement, as we illustrate in three examples. It is an alternative to the existing procedures based on a formulation in discrete time (Gautier, 2004) or to numerical methods (Collard et al., 2008), which are well suited when the characteristic functions are complicated.

Most importantly, the theoretical analysis of MFDEs gives rise to new and interesting dynamic configurations and reconsiders the link between the local uniqueness of a solution and the stability of a steady state. In particular, we show that the dynamics of a predetermined variable may be both stable and indeterminate, while with an ODE stability implies uniqueness. In addition, the dynamics of a non-predetermined variable may be both stable and determinate, while with an ODE stability implies indeterminacy. Finally, we show that a non-predetermined variable does not generally jump to its steady state value.

The article is organized as follows. In Section 2, we begin with a simple economic model illustrating the equations we are going to study. Then, we explain why the presence of advances and the definition of initial conditions imply that the mathematical problem is ill-posed, and why this may lead to the non-existence or the multiplicity of solutions. This section also allows us to relate our contribution to existing literature. In Section 3, we present our results on the existence and uniqueness of solutions of MFDEs as well as two examples that we solve in order to illustrate our theorems. In Section 4, we extend our results to algebraic equations of mixed type and solve an example. We also put forward a linearization theorem in order to apply our results to non-linear MFDEs. We conclude in Section 5.

2. Presentation of the problem

2.1. A simple economic model

To introduce the equations we are going to study, let us consider a model where the investment goods follow a "one-hoss shay" depreciation rule. Such goods contribute to the capital stock throughout their lifetime before falling to a zero scrap value. Let $I(s)$ be the investment implemented at date $s \leq t$ and let $\tau \in \mathbb{R}_+$ be the lifespan of the investment goods. The capital stock is therefore at date t equal to the sum of all investments made between dates $t-\tau$ and t:

$$K(t) = \int_{t-\tau}^{t} I(s)\, ds. \tag{1}$$

If one considers both a Solowian framework, where the investment chosen at time t is proportional to the demand received by the firms in the same period (i.e. $I(t) = \alpha Y(t)$), and an equilibrium on the goods market that equalizes demand and production such that $Y(t) = F(K(t))$, Eq. (1) can be rewritten as:

$$K(t) = \alpha \int_{t-\tau}^{t} F(K(s))\, ds. \tag{2}$$

Differentiating with respect to time, one obtains a DDE:

$$K'(t) = \alpha F(K(t)) - \alpha F(K(t-\tau)). \tag{3}$$

If one considers a more sophisticated framework where the investment chosen at time t is proportional to the demand that firms expect to receive throughout the lifetime of the investment good, Eq. (1) can be rewritten as:

$$K(t) = \alpha \int_{t-\tau}^{t} \int_{s}^{s+\tau} F(K(v))\, dv ds, \tag{4}$$

which is an algebraic equation of mixed type. Differentiating (4) with respect to time yields an MFDE:

$$K'(t) = \alpha \int_{t}^{t+\tau} F(K(v))\, dv - \alpha \int_{t-\tau}^{t} F(K(v))\, dv. \tag{5}$$

In Section 4.1.3, we solve Eq. (4) in the particular case of a linear production function given by $F(K(t)) = AK(t)$. This is, of course, an illustrative example, as MFDEs usually arise in models with more relevant microeconomic foundations. But in all cases, delays are due to the vintage structure of a stock variable (e.g. capital, population, price or wage contracts, etc.) whereas advances are due to the forecasts made by the agents. Comprehensive presentations of the use of MFDEs in economics are given in Collard et al. (2008) for vintage capital models and in d'Albis and Augeraud-Véron (2011) for overlapping generation models.

2.2. The mathematical problem

Let $t \in \mathbb{R}_+$ be the time index. We consider the following scalar linear MFDE:

$$x'(t) = \int_{-a}^{b} x(u+t)\, d\mu(u), \tag{6}$$

where $(a, b) \in \mathbb{R}_+^2$ and where μ is a measure on $[-a, b]$. Eq. (6) characterizes dynamics for which the time-derivative at time t of a variable x, denoted $x'(t)$, depends on both the delayed values of x over the interval $[t-a, t)$ and on the advanced values of x over the interval $(t, t+b]$.

Due to the delay, initial conditions are defined over an interval. But, as usual, initial conditions may be of two different types. First, variable x can be predetermined (sometimes referred to as backward-looking), with initial condition of Eq. (6) written as:

$$x(\xi) = x_0(\xi) \quad \text{for } \xi \in [-a, 0], \tag{7}$$

where $x_0 \in \mathcal{C}([-a, 0])$, the space of continuous functions on $[-a, 0]$. Second, for a non-predetermined (or forward-looking) variable, the initial condition can be written as:

$$x(\xi) = x_0(\xi) \quad \text{for } \xi \in [-a, 0), \tag{8}$$

where $x_0 \in \mathcal{C}^b([-a, 0])$, the space of continuous functions on $[-a, 0)$ such that $x_0(0^-)$ exists. Here, $x(0^+)$ is not given and may be different from $x_0(0^-)$.

Note that Eq. (6) admits a unique steady state, namely $x = 0$, but may also be solved by functions that grow at a constant growth rate, thereby exhibiting a balanced growth path (BGP). This allows us to characterize the local stability of either a steady state or a BGP.

Let us first define a solution to the problem being considered.