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Bubbles and trading in incomplete markets☆

Camelia Bejan[a], Florin Bidian[b,*]

[a] School of Business, University of Washington-Bothell, 18115 Campus Way NE, Bothell, WA 98011, United States

[b] Robinson College of Business, Georgia State University, RMI, PO Box 4036, Atlanta, GA 30302-4036, United States

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ABSTRACT

We show that an intrinsic property of a large class of rational bubbles is their capacity to relax the agents' debt limits. Any bubble that preserves the set of pricing kernels, or equivalently, the asset span, has effectively an identical effect on consumption and real interest rates as an appropriate relaxation of debt limits, proportional to the size of the bubble. Thus the collapse of a bubble amounts to a contraction of agents' debt limits, and conversely, a bubble can arise to supplement the credit available in the economy.

1. Introduction

Episodes of large stock market run-ups followed by abrupt crashes, without matching movements in fundamentals, are referred to as bubbles. Formally, a (rational) bubble is defined as the price of an asset in excess of its fundamental value, computed as the discounted (at market rates) present value of dividends.

We show that a large class of rational bubbles are equivalent, from the point of view of consumption and real interest rates, to a relaxation of agents' debt limits. An equilibrium (under some fixed credit limits) with bubbles in the prices of some assets allows agents the same level of consumption they would get in a no-bubble equilibrium of an alternative economy with more *relaxed* credit limits.

We build on the insight of Kocherlakota (2008), who showed that arbitrary discounted (by the pricing kernel) positive martingales can be introduced into asset prices as bubbles, while leaving agents' consumption and the pricing kernel unchanged, as long as the debt limits of the agents are allowed to be adjusted upwards (that is, tightened) by their initial endowment of the assets multiplied with the bubble term. In other words, although at the bubbly equilibrium agents are subject to tighter debt limit, they can still enjoy the same level of consumption they would under more relaxed debt limits (and no bubbles in the asset prices). In that sense, bubbles are equivalent to a relaxation of debt limits. The modified debt limits bind in exactly the same dates and states. Kocherlakota (2008) refers to this result as the "bubble equivalence theorem", and to this technique of introducing bubbles as "bubble injections".

At the heart of the argument is that the introduction of a bubble gives consumers a windfall, proportional to their initial holding of the asset, which can be sterilized, leaving their budgets unaffected, by an appropriate tightening of the debt limits. Conversely, the pricking of a bubble and the resulting drop in agents' wealth can be compensated by a relaxation of debt limits.

A major limitation of Kocherlakota (2008) result is the assumption that agents can trade in a full set of state-contingent claims to consumption next period, in addition to the existing long-lived securities. Hence one might infer that the bubble equivalence theorem is associated to knife-edge situations, and that it might not apply to incomplete markets environments or even to economies with dynamically complete markets (rather than Arrow–Debreu complete).

We prove that a version of the bubble equivalence theorem holds even when markets are incomplete, or only dynamically complete. The equivalence has two parts. The *bubble injection* direction characterizes completely the set of processes that can be injected as bubbles in asset prices through a tightening of debt limits, while preserving the real variables. Such processes are called *pricing kernel-preserving*, or *kernel-preserving*, for short. The reverse direction, or the *bubble pricking* direction, shows that a large class of bubbles (those that are kernel-preserving) can be pricked and result in identical real variables, as long as agents' debt limits are relaxed.

The kernel-preserving processes, as the name suggests, are those nonnegative processes that result in an identical set of pricing kernels if added to (bubble-free) asset prices, or conversely,

☆ This paper is based on Chapter 3 in Bidian (2011), "Payout Policy and Bubbles". Parts of it have been written while Camelia Bejan was at Rice University. It was also circulated under the longer name "Limited Enforcement, Bubbles and Trading in Incomplete Markets.".

* Corresponding author.
E-mail addresses: cameliab@uw.edu (C. Bejan), fbidian@gsu.edu (F. Bidian).

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if subtracted from (bubbly) asset prices. Equivalently, they are discounted martingales (under some pricing kernel) that preserve the asset span (if added to bubble-free prices, or deducted from bubbly prices). A kernel-preserving process is also a martingale when discounted by any pricing kernel associated to the initial prices or any pricing kernel associated to the prices inflated by the process. In particular, any nonnegative process which equals the value of a self-financing trading strategy will generically be a kernel-preserving process.

Our results show that the setup of Kocherlakota (2008) with Arrow-complete markets and additional, redundant long-lived assets (rather than dynamically complete markets) is not innocuous. In his framework, the pricing kernel with or without a bubble (in the long-lived assets) is the same and is uniquely pinned down by the prices of the Arrow securities. With dynamically complete markets, the injection or the pricking of a bubble can distort the asset span and the pricing kernel, and not lead to an equivalent equilibrium.

The bubble equivalence theorem has additional appeal in environments with endogenous debt limits, as in Alvarez and Jermann (2000). In these models, agents have the option to default on debt and receive a predetermined continuation utility, and the markets (competitive financial intermediaries) select the largest debt limits so that repayment is always individually rational given future bounds on debt. It turns out that both the debt limits of the bubble-free equilibrium and the tighter debt limits of the equivalent bubbly equilibrium are the endogenous bounds allowing for maximal credit expansion and preventing default. We allow for more general punishments after default than in Kocherlakota (2008). In particular, we cover the case where upon default the agents are forbidden to carry debt (Bulow and Rogoff, 1989; Hellwig and Lorenzoni, 2009). Therefore the "incomplete markets" in the title of the paper refers to both environments with exogenously, respectively endogenously incomplete (due to limited enforcement) markets.

It is easy to misinterpret the injection direction of the bubble equivalence theorem as a "license" to create bubbles freely. However, bubbles in positive supply assets cannot exist in economies with *high interest rates*, that is with finite present value of aggregate consumption (Santos and Woodford, 1997; Werner, 2014; Kocherlakota, 1992), as long as agents are not prevented from reducing their share holdings, that is if their debt limits are nonpositive. Intuitively, bubbles grow on average at the rate of interest rates. With high interest rates, the bubble must become very large relative to aggregate endowment, even if this happens with small probability. But this is incompatible with the presence of optimizing, forward looking agents, who do not allow their financial wealth to exceed the present value of their future consumption. As shown by Bidian (2011, Chapter 2) and Bidian (2014a), this argument against the existence of bubbles in economies with high interest rates is extremely robust and applies to environments with asymmetric information, heterogeneous beliefs and quite general portfolio constraints.

Therefore with *high* interest rates, the tighter debt bounds needed to sterilize the wealth effects of a bubble injection in an asset in positive supply must be positive at some dates and states (even though this may happen with arbitrarily small probability). However, if an asset is in zero supply, and initially none of the agents hold any shares, a bubble injection has no wealth or allocational effects. One can inject "freely" any (nonnegative) kernel-preserving process as a bubble into the price of that asset, while preserving real interest rates and agents' consumption and debt limits.

Low interest rates arise naturally with the enforcement limitations studied in Section 4, since in equilibrium the interest rates adjust to a lower level to entice agents to repay their debt and prevent default. Hellwig and Lorenzoni (2009) (see also Werner, 2014) show that if the penalty for default is an *interdiction to borrow*, then all non-autarchic equilibria must in fact have low interest rates, and bubble injections with nonpositive debt limits are possible. Bidian (2011, Chapter 4) and Bidian (2014b) show that low interest rates can arise in equilibrium and that bubble injections with nonpositive debt limits are possible for the other common penalties for default encountered in the literature: a *permanent* or a *temporary interdiction to trade* after default. All the mentioned examples of bubble injections with nonpositive debt limits feature complete markets.

The bubble pricking direction of Theorem 3.3 shows that the intrinsic feature of kernel-preserving bubbles is to relax financial constraints. Such bubbles must be *unambiguous*, in that they do not vanish if the present value of dividends (fundamental value) is calculated using any valid pricing kernel. It follows that, for general environments with incomplete markets, an unexpected collapse of a kernel-preserving bubble would not affect agents' consumption if their debt limits are relaxed by an amount proportional to the size of the bubble. In the absence of such an increase in the availability of credit, a bubble collapse amounts to a credit crunch, and therefore can be contractionary (see, for example, Guerrieri and Lorenzoni, 2011).

Therefore kernel-preserving bubbles act as devices that relax agents' debt limits. A host of recent papers point out similarly, but in very specific environments, that bubbles can arise in the presence of financial frictions, and help relax the underlying borrowing constraints (Kocherlakota, 2009; Martin and Ventura, 2012; Giglio and Severo, 2012; Farhi and Tirole, 2012). These bubbles facilitate the transfer of resources from unproductive entrepreneurs to the productive ones, by increasing the borrowing capacity of the latter. Miao and Wang (2011) make a related point, but they emphasize the multiplicity of equilibria in economies with limited enforcement, studied also in Hellwig and Lorenzoni (2009) and Bidian (2014b). In their model, bubbles are defined as the difference between the value of the firm and the value predicted using the q theory of investment. These papers analyze the production sector, shutting down (non-entrepreneurs) consumers from borrowing and lending. By contrast, we intentionally focus squarely on the consumer sector, allowing consumers to borrow and lend to each other, in a Bewley–Aiyagari environment.

2. Model

Time periods are indexed by the set $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ and by the filtration $(\mathcal{F}_t)_{t=0}^{\infty}$, which is an increasing sequence of finite partitions $\mathcal{F}_t \subset \mathcal{F}$ on the set of states of the world Ω with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We interpret $\mathcal{F}_t$ as the information available at period t.

Let X be the set of all stochastic processes adapted to $(\mathcal{F}_t)_{t=0}^{\infty}$,[1] and denote by X_+ (respectively X_{++}) the processes $x \in X$ such that $x_t \geq 0$ P-almost surely (respectively $x_t > 0$ P-almost surely) for all $t \in \mathbb{N}$. All statements, equalities, and inequalities involving random variables are assumed to hold only "P-almost surely", and we will omit adding this qualifier. When $K, L \in \mathbb{N} \setminus \{0\}$, let $X^{K\times L}$, respectively $X_+^{K\times L}$ be the set of vector (or matrix) processes $(y^{ij})_{1\leq i\leq K, 1\leq j\leq L}$ with $y^{ij} \in X$, respectively $y^{ij} \in X_+$.

There is a single consumption good and a finite number, I, of consumers. An agent $i \in \{1, 2, \ldots, I\}$ has preferences represented by a utility $U : X_+ \to \mathbb{R}$ given by $U^i(c) = E\sum_{t=0}^{\infty} u_t^i(c_t^i)$, where c_t^i is the consumption of i, and $E(\cdot)$ is the expectation operator with respect to probability P. The per-period utility $u_t^i : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing. The conditional expectation given

[1] This is the set of sequences $x = (x_t)_{t\in\mathbb{N}}$ of random variables $x_t : \Omega \to \mathbb{R}$ such that x_t is $\mathcal{F}_t$-measurable.