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The m-step, same-step, and any-step competition graphs

Wei Ho

Princeton University, Department of Mathematics, Princeton, NJ 08544, USA

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Abstract

The competition graph and its generalizations have interested graph theorists for over 30 years. Recently, Cho, Kim, and Nam introduced the *m*-step competition graph and computed the 2-step competition numbers of paths and cycles. We extend their results in a partial determination of the *m*-step competition numbers of paths and cycles. In addition, we introduce two new variants of competition graphs: same-step and any-step. We classify same-step and any-step competition graphs and investigate their related competition numbers.

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Keywords: Competition graph; Competition number; *m*-step competition graph; Same-step competition graph; Any-step competition graph

1. Introduction

Motivated by the study of ecological systems, Cohen [4] in 1968 introduced the concept of a competition graph of a digraph. In ecological terms, the digraph represents a food web, in which the vertices represent species and arcs point from predators to their prey. We define the *competition graph* $C(D)$ of a digraph D as having the same vertex set as D and having an edge between vertices v and w if and only if there exists some vertex x such that (v, x) and (w, x) are arcs of D. We call x a *common prey* of vertices v and w. The *competition*

E-mail address: who@math.princeton.edu.

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number $k(G)$ of a graph G is the minimum number of isolated vertices such that the disjoint union of G and these isolated vertices is the competition graph of an acyclic digraph.

Note that in the definition of a competition graph, the digraph can be arbitrary; for determining a competition number of a graph, however, we are restricted to *acyclic* digraphs. This distinction will remain for variants of competition graphs.

Since 1968, several variants of competition graphs have been studied, including the common enemy graph in [9], the competition-common enemy graph in [11], the niche graph in [2], the p-competition graph in [8], and the competition multigraph in [1]. Cho et al. [3] introduced the m-step competition graph in 2000. In the m-step competition graph of a digraph, an edge exists between two vertices if both have directed m-step walks to a third vertex in the digraph. The concept of the competition number generalizes naturally to the m-step competition number, the minimum number of isolated vertices that need to be added to a graph to make it the m-step competition graph of an acyclic digraph.

Cho et al. [3] determine 2-step competition numbers for paths and cycles. In Section 2 of this paper, we first extend their results to m-step competition numbers for these families of graphs. Then in Sections 3 and 4, respectively, we introduce the same-step and any-step competition graphs and prove results on the classification of these graphs and their competition numbers. Finally, in Section 5, we discuss open questions concerning these variants of competition graphs and competition numbers.

2. The *m*-step competition numbers of paths and cycles

We first define the m-step competition graph and related concepts more formally.

Given a digraph D and a positive integer m, suppose there exists a directed m-step walk from vertex v to vertex x. Then we call v an *m-step predator* of x, and x an *m-step prey* of v. If vertices v and w both are m-step predators of x, then we call x a *common m-step prey* of v and w. We define the m-step competition graph $C^m(D)$ of a digraph D as the graph with the same vertex set as D and an edge between vertices v and w if and only if they have a common m-step prey.

Given a positive integer m, the m-step digraph D^m of a digraph D is the digraph with the same vertex set as D and an arc (u, v) if and only if there exists a directed walk of length m from u to v in D.

Recall that we can represent a simple digraph on n vertices by an $n \times n$ adjacency matrix with boolean entries. Since $C^m(D) = C(D^m)$ [3], we have the following proposition:

Proposition 2.1. *If A is the adjacency matrix of a digraph D, then the adjacency matrix of $C^m(D)$ is given by $A^m(A^T)^m$.*

Given an arbitrary graph G, we can add some number of isolated vertices to G to obtain an m-step competition graph of an acyclic digraph. In the next two paragraphs, we provide such a construction in order to motivate the definition of the m-step competition number $k^{(m)}(G)$ of a graph G.