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# Efficient parallel recognition of cographs

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**Abstract**

In this paper, we establish structural properties for the class of complement reducible graphs or cographs, which enable us to describe efficient parallel algorithms for recognizing cographs and for constructing the cotree of a graph if it is a cograph; if the input graph is not a cograph, both algorithms return an induced $P_4$. For a graph on $n$ vertices and $m$ edges, both our cograph recognition and cotree construction algorithms run in $\mathrm{O}(\log^2 n)$ time and require $\mathrm{O}((n+m)/\log n)$ processors on the EREW PRAM model of computation. Our algorithms are motivated by the work of Dahlhaus (Discrete Appl. Math. 57 (1995) 29–44) and take advantage of the optimal $\mathrm{O}(\log n)$-time computation of the co-connected components of a general graph (Theory Comput. Systems 37 (2004) 527–546) and of an optimal $\mathrm{O}(\log n)$-time parallel algorithm for computing the connected components of a cograph, which we present. Our results improve upon the previously known linear-processor parallel algorithms for the problems (Discrete Appl. Math. 57 (1995) 29–44; J. Algorithms 15 (1993) 284–313): we achieve a better time-processor product using a weaker model of computation and we provide a certificate (an induced $P_4$) whenever our algorithms decide that the input graphs are not cographs.


*Keywords:* Perfect graphs; Cographs; Cotree; Connected and co-connected components; Parallel algorithms; Parallel recognition; Certifying algorithms

## 1. Introduction

The *complement reducible graphs*, also known as *cographs*, are defined as the class of graphs formed from a single vertex under the closure of the operations of union and complementation. More precisely, the class of cographs is defined recursively as follows:

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(i) a single-vertex graph is a cograph, (ii) the disjoint union of cographs is a cograph and (iii) the complement of a cograph is a cograph.

Cographs have arisen in many disparate areas of applied mathematics and computer science and have been independently rediscovered by various researchers under various names such as $D^*$-graphs [16], $P_4$ restricted graphs [8,9], 2-parity graphs and Hereditary Dacey graphs or HD-graphs [24]. Cographs are perfect and in fact form a proper subclass of permutation graphs and distance hereditary graphs; they contain the class of quasi-threshold graphs and, thus, the class of threshold graphs [5,11]. Furthermore, cographs are precisely the graphs which contain no induced subgraph isomorphic to a $P_4$ (chordless path on four vertices).

Cographs were introduced in the early 1970s by Lerchs [18] who studied their structural and algorithmic properties. Along with other properties, Lerchs has shown that the class of cographs coincides with the class of $P_4$ restricted graphs, and that the cographs admit a unique tree representation, up to isomorphism, called a *cotree*. The cotree of a cograph $G$ is a rooted tree such that:

(i) each internal node, except possibly for the root, has at least two children;
(ii) the internal nodes are labelled by either 0 (0-*nodes*) or 1 (1-*nodes*); the children of a 1-node (0-node resp.) are 0-nodes (1-nodes, resp.), i.e., 1- and 0-nodes alternate along every path from the root to any node of the cotree;
(iii) the leaves of the cotree are in a 1-to-1 correspondence with the vertices of $G$, and two vertices $v_i, v_j$ are adjacent in $G$ if and only if the least common ancestor of the leaves corresponding to $v_i$ and $v_j$ is a 1-node.

Lerchs' definition required that the root of a cotree be a 1-node; if, however, we relax this condition and allow the root to be a 0-node as well, then we obtain cotrees whose internal nodes all have at least two children, and whose root is a 1-node if and only if the corresponding cograph is connected.

There are several recognition algorithms for the class of cographs. Sequentially, linear-time algorithms for recognizing cographs were given in [9,6]. In a parallel setting, cographs can be efficiently (but not optimally) recognized in polylogarithmic time using a polynomial number of processors. Adhar and Peng [1] described a parallel algorithm for this problem which, on a graph on $n$ vertices and $m$ edges, runs in $O(\log^2 n)$ time and uses $O(nm)$ processors on the CRCW PRAM model of computation. Another recognition algorithm was developed by Kirkpatrick and Przytycka [17], which requires $O(\log^2 n)$ time with $O(n^3/\log^2 n)$ processors on the CREW PRAM model. Lin and Olariu [19] proposed an algorithm for the recognition and cotree construction problem which requires $O(\log n)$ time and $O((n^2+nm)/\log n)$ processors on the EREW PRAM model. Recently, Dahlhaus [10] proposed a nearly optimal parallel algorithm for the same problem which runs in $O(\log^2 n)$ time with $O(n+m)$ processors on the CREW PRAM model. Another cograph recognition and cotree construction algorithm was presented by He [12]; it requires $O(\log^2 n)$ time and $O(n+m)$ processors on the CRCW PRAM model.

Since the cographs are perfect, many interesting optimization problems in graph theory, which are NP-complete in general graphs, have polynomial sequential solutions and