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Note

Note on: N.E. Aguilera, M.S. Escalante, G.L. Nasini, "The disjunctive procedure and blocker duality"☆

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Abstract

Aguilera et al. [Discrete Appl. Math. 121 (2002) 1–13] give a generalization of a theorem of Lehman through an extension $\bar{P}_j$ of the disjunctive procedure defined by Balas, Ceria and Cornuéjols. This generalization can be formulated as

(A) For every clutter $\mathscr{C}$, the disjunctive index of its set covering polyhedron $\mathscr{Q}(\mathscr{C})$ coincides with the disjunctive index of the set covering polyhedron of its blocker, $\mathscr{Q}(b(\mathscr{C}))$.

In Aguilera et al. [Discrete Appl. Math. 121 (2002) 1–3], (A) is indeed a corollary of the stronger result

(B) $\bar{P}_J([\bar{P}_J(\mathscr{Q}(\mathscr{C}))]^B) = [\mathscr{Q}(\mathscr{C})]^B$.

Motivated by the work of Gerards et al. [Math. Oper. Res. 28 (2003) 884–885] we propose a simpler proof of (B) as well as an alternative proof of (A), independent of (B). Both of them are based on the relationship between the "disjunctive relaxations" obtained by $\bar{P}_j$ and the set covering polyhedra associated with some particular minors of $\mathscr{C}$.

Keywords: Disjunctive procedure; Blocking type polyhedra; Blocker duality; Clutter

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In this note we work with the notation and concepts used in [1] and add some other specific notation. Let us first recall some definitions and known results.

Given $U \subset \mathbf{R}^n$, conv(U) denotes the convex hull of the elements of U. For a polyhedron $\mathcal{Q} \subset \mathbf{R}^n$, $\mathcal{Q}^*$ denotes the polyhedron

$$\mathcal{Q}^* = \mathrm{conv}(\mathcal{Q} \cap \mathbf{Z}^n).$$

A polyhedron $\mathcal{Q}$ is of *blocking type* if $\mathcal{Q} \subset \mathbf{R}^n_+$ and if $y \geqslant x \in \mathcal{Q}$ implies $y \in \mathcal{Q}$. It is known that if $\mathcal{Q}$ is of blocking type, then $\mathcal{Q}^*$ also is. The *blocker* $\mathcal{Q}^{\mathrm{B}}$ of a blocking type polyhedron $\mathcal{Q}$, is defined by

$$\mathcal{Q}^{\mathrm{B}} = \{\pi \in \mathbf{R}^n_+ : \pi \cdot x \geqslant 1, \text{ for all } x \in \mathcal{Q}\},$$

and it is known that $(\mathcal{Q}^{\mathrm{B}})^{\mathrm{B}} = \mathcal{Q}$.

In [1], Aguilera et al. work on [0, 1] blocking type polyhedra, i.e. blocking type polyhedra with extreme points in $[0,1]^n$. For $j \in \{1, ..., n\}$, they define a disjunctive procedure over $\mathcal{Q}$ as

$$\bar{P}_j(\mathcal{Q}) = P_j(\mathcal{Q}_0) + \mathbf{R}^n_+,$$

where $\mathcal{Q}_0 = \mathcal{Q} \cap [0,1]^n$ and P_j denotes the procedure defined by Balas, Ceria and Cornuéjols.

This new procedure satisfies similar properties as those of P_j, all of them proved in [1]. In particular, given $J = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$, since applying iteratively the procedure does not depend on the order, $\bar{P}_{i_1}(\bar{P}_{i_2}(...(\bar{P}_{i_k}(\mathcal{Q}))))$ can be denoted by $\bar{P}_J(\mathcal{Q})$.

Given $R \subset \{1, \ldots, n\}$, let

$$\mathscr{H}_R = \{x \in \mathbb{R}^n_+ : x_i = 0 \text{ for } i \in R\} \quad \text{and} \quad \overline{\mathscr{G}}_R = \{x \in \mathbb{R}^n_+ : x_i \geqslant 1 \text{ for } i \in R\}.$$

Throughout the rest of this note, J will be a fixed subset of $\{1, \ldots, n\}$ and we will write any $x \in \mathbb{R}^n$ as $x = (\overline{x}, \widetilde{x})$, where $\overline{x} \in \mathbb{R}^{|J|}$ and $\widetilde{x} \in \mathbb{R}^{n-|J|}$.

Given $R \subset J$ and $\mathcal{Q}$ a [0, 1] blocking type polyhedron, we denote $\overline{R} = J \backslash R$ and $\mathcal{Q}_R = \mathcal{Q} \cap \mathscr{H}_{\overline{R}} \cap \overline{\mathscr{G}}_R$.

In [1] it is proved that

$$\bar{P}_J(\mathcal{Q}) = \mathrm{conv}\left(\bigcup_{\mathrm{R} \subset \mathrm{J}} \mathcal{Q}_R\right). \tag{1}$$

Hence

$$\mathcal{Q}^* \subset \bar{P}_J(\mathcal{Q}) \subset \mathcal{Q} \quad \text{and} \quad \bar{P}_{\{1,\ldots,n\}}(\mathcal{Q}) = \mathcal{Q}^*. \tag{2}$$

This last equality allows talking about the minimum number of iterations needed so as to find $\mathcal{Q}^*$, which is called the *disjunctive index* of $\mathcal{Q}$. It is clear that $\mathcal{Q}^* = \mathcal{Q}$ if and only if the disjunctive index of $\mathcal{Q}$ is zero.

From (1) every extreme point of $\bar{P}_J(\mathcal{Q})$ is an extreme point of $\mathcal{Q}_R$, for some $R \subset J$. Let x^R denote, for any $R \subset J$, the point in $\{0,1\}^{|J|}$ with $x^R_i = 1$ if and only if $i \in R$. We have the following:

Lemma 1. *Let $x = (\overline{x}, \widetilde{x}) \in \mathbb{R}^n_+$. If x is an extreme point of $\mathcal{Q}_R$ then $\overline{x} = x^R$.*