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The many benefits of putting stack filters into disjunctive or conjunctive normal form

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Abstract

Stack filters are nonlinear filters used for image processing (examples: median filters, order statistics). In the translation-invariant case a stack filter is determined by a positive Boolean function b. Many important properties of stack filters (idempotency, co-idempotency, order relations) can be tested in polynomial time if the DNF and/or CNF of b are known.

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1. Introduction

Let us go into medias res. One simple example of a stack filter would be the operator $\Phi : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ which maps a series $f = \{f_i \mid i \in \mathbb{Z}\}$ to the series Φf whose ith component is defined by $[\Phi f]_i := (f_{i-2} \wedge f_i) \vee f_{i+1}$. Hereby $f_i \wedge f_j$ and $f_i \vee f_j$ are defined as the minimum and maximum, of the real numbers f_i and f_j, respectively. Not surprisingly, the behaviour of Φ is determined by the underlying positive Boolean function $b : \{0,1\}^4 \to \{0,1\}$ that maps $(x_{-2}, x_{-1}, x_0, x_1)$ to $(x_{-2} \wedge x_0) \vee x_1$.

In Section 2 we review the conjunctive (CNF) and disjunctive (DNF) normal forms of positive Boolean functions and, for later purposes, explicitly derive one from the other for some nontrivial $b_n : \{0,1\}^n \to \{0,1\}$.

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In Section 3 it is indicated how stack filters $\Psi : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ arise in nonlinear image processing. Interestingly, Ψ need not originally be defined in terms of $\wedge$ and $\vee$. We then proceed to the computation of the DNF and CNF of some concrete stack filters (i.e., of their underlying positive Boolean functions). In particular the b_n of Section 2 corresponds to the stack filter $\Psi := L_n \circ U_n$ where L_n and U_n are the thoroughly investigated stack filters of [6].

In Section 4 we discuss four "benefits" of normal forms of stack filters. As to the first benefit, when both the CNF and DNF of Ψ are known, there is a polynomial algorithm [10] to decide whether or not Ψ is *idempotent*, i.e. whether $\Psi \circ \Psi = \Psi$. Second, the *co-idempotency* of Ψ, i.e. $(I - \Psi) \circ (I - \Psi) = I - \Psi$, where I is the identity map, can also be tested in polynomial time. We further expand upon the related computation of all *noise* series $g := f - \Psi f$ of Ψ, in particular for $\Psi := L_n \circ U_n$. Third, two stack filters Φ and Ψ are said to be comparable, say $\Phi \leqslant \Psi$, if $\Phi f \leqslant \Psi f$ for all series $f \in \mathbb{R}^{\mathbb{Z}}$. Given their DNF (or CNF) it can be tested in polynomial time whether or not $\Phi \leqslant \Psi$. Fourth, a stack filter Φ is *neighbourly trend preserving* if $f_i \leqslant f_{i+1}$ implies $[\Phi f]_i \leqslant [\Phi f]_{i+1}$, and $f_i \geqslant f_{i+1}$ implies $[\Phi f]_i \geqslant [\Phi f]_{i+1}$. If Φ is given in normal form this property can be checked in polynomial time.

2. Prerequisites about positive Boolean functions

Let us review some well-known facts from Boolean logic which shall be crucial in later sections. For x, y in $\{0,1\}^n$ write $x \leqslant y$ if $x_i \leqslant y_i$ for all $1 \leqslant i \leqslant n$. Any function $b : \{0,1\}^n \to \{0,1\}$ is called a *Boolean function*. It is *positive* (or *monotone*) if for all $x, y \in \{0,1\}^n$ it follows from $x \leqslant y$ that $b(x) \leqslant b(y)$. As opposed to the general case, a positive b admits a unique *minimal disjunctive normal form* (*the* DNF), and dually a unique *minimal conjunctive normal form* (the CNF).

Namely, for all $x = (x_1, \ldots, x_n)$ in $\{0,1\}^n$ put $One(x) := \{i \mid x_i = 1\}$ and $Zero(x) := \{i \mid x_i = 0\}$. A subset $C \subseteq \{1, \ldots, n\}$ is a 1-*set* of b if $b(x) = 1$ for the unique x with $One(x) = C$. Dually call $D \subseteq \{1, \ldots, n\}$ a 0-*set* of b if $b(y) = 0$ for the unique y with $Zero(y) = D$. Let $\mathscr{C} = \mathscr{C}(b)$ be the set of all nonvoid minimal 1-sets and let $\mathscr{D} = \mathscr{D}(b)$ be the set of all nonvoid minimal 0-sets.[1] If $b(x) = 1$ for all $x \in \{0,1\}^n$ then $\mathscr{D} = \emptyset$. Dually, if $b(x) = 0$ for all $x \in \{0,1\}^n$ then $\mathscr{C} = \emptyset$. But for a nonconstant positive Boolean function b both clusters $\mathscr{C}$ and $\mathscr{D}$ are nonvoid antichains. (A family of sets is an *antichain* if no member properly contains another member of that family.) The DNF (respectively the CNF) of b is then defined as

$$\bigvee_{C \in \mathscr{C}} \left(\bigwedge_{i \in C} x_i \right) \left(\text{respectively} \quad \bigwedge_{D \in \mathscr{D}} \left(\bigvee_{j \in D} x_j \right) \right). \tag{1}$$

[1] Other authors speak of *T*-sets and *F*-sets, rather than of 1-sets and 0-sets of a Boolean function. While their *T*-sets coincide with our 1-sets, their *F*-sets are usually defined to be the *complements* within $[1, n]$ of our 0-sets.