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Weakly useful sequences

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Abstract

An infinite binary sequence x is defined to be

(i) *strongly useful* if there is a computable time bound within which *every* decidable sequence is Turing reducible to x; and
(ii) *weakly useful* if there is a computable time bound within which all the sequences in a non-measure 0 subset of the set of decidable sequences are Turing reducible to x.

Juedes, Lathrop, and Lutz [Theorectical Computer Science 132 (1994) 37] proved that every weakly useful sequence is strongly deep in the sense of Bennett [The Universal Turing Machine: A Half-Century Survey, 1988, 227] and asked whether there are sequences that are weakly useful but not strongly useful. The present paper answers this question affirmatively. The proof is a direct construction that combines the *martingale*

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diagonalization technique of Lutz [9] with a new technique, namely, the construction of a sequence that is "computably deep" with respect to an arbitrary, given uniform reducibility. The *abundance* of such computably deep sequences is also proven and used to show that every weakly useful sequence is computably deep with respect to every uniform reducibility.

Keywords: Computability; Randomness; Random sequence; Computational depth; Logical depth; Computable measure; Resource-bounded measure; Useful; Weakly useful

1. Introduction

It is a truism that the usefulness of a data object does not vary directly with its information content. For example, consider two infinite binary strings, χ_K, the characteristic sequence of the halting problem (whose nth bit is 1 if and only if the nth Turing machine halts on input n), and z, a sequence that is algorithmically random in the sense of Martin-Löf [10]. The following facts are well-known.

(1) The first n bits of χ_K can be specified using only $O(\log n)$ bits of information, namely, the *number* of 1's in the first n bits of χ_K [1].
(2) The first n bits of z cannot be specified using significantly fewer than n bits of information [10].
(3) Oracle access to χ_K would enable one to decide *any* decidable sequence in polynomial time (i.e., decide the nth bit of the sequence in time polynomial in the length of the binary representation of n) [11].
(4) Even with oracle access to z, most decidable sequences cannot be computed in polynomial time. (This appears to be folklore, known at least since [2].)

Facts (1) and (2) tell us that χ_K contains far less information than z. In contrast, facts (3) and (4) tell us that χ_K is computationally much more useful than z. That is, the information in χ_K is "more usefully organized" than that in z.

Bennett [2] introduced the notion of *computational depth* (also called "logical depth") in order to quantify the degree to which the information in an object has been organized. In particular, for infinite binary sequences, Bennett defined two "levels" of depth, *strong depth* and *weak depth*, and argued that the above situation arises from the fact that χ_K is strongly deep, while z is not even weakly deep. (The present paper is motivated by the study of computational depth, but does not directly use strong or weak depth, so definitions are omitted here. The interested reader is referred to [2,5,7] for details, and for related aspects of algorithmic information theory.)

Investigating this matter further, Juedes et al. [5] considered two "levels of usefulness" for infinite binary sequences. Specifically, let $\mathbf{C}$ be the Cantor space of all infinite binary sequences and let DEC be the set of all decidable elements of $\mathbf{C}$. For $x \in \mathbf{C}$ and $t\colon \mathbb{N} \to \mathbb{N}$, let $\mathrm{DTIME}^x(t)$ be the set of all $y \in \mathbf{C}$ for which there exists an oracle Turing machine M that, on input $n \in \mathbb{N}$ with oracle x, computes $y[n]$, the nth bit of y, in at most $t(\ell)$ steps, where ℓ is the number of bits in the binary representation of n. Then a sequence $x \in \mathbf{C}$ is defined to be *strongly useful* if there is a computable time bound $t\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{DTIME}^x(t)$ contains *every* decidable sequence. A sequence $x \in \mathbf{C}$