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The converse of a stochastic relation ☆

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Abstract

Transition probabilities are proposed as the stochastic counterparts to set-based relations. We propose the construction of the converse of a stochastic relation. It is shown that two of the most useful properties carry over: the converse is idempotent as well as anticommutative. The nondeterminism inherent in a stochastic relation is defined and briefly investigated. We define a bisimulation relation, and indicate conditions under which this relation is transitive; moreover it is shown that bisimulation and converse are compatible.

Keywords: Stochastic relations; Concurrency; Bisimulation; Converse; Relational calculi; Nondeterminism

1. Introduction

The use of relations is ubiquitous in Mathematics, Logic and Computer Science, their systematic study goes back as far as Schröder's seminal work [21]. Ongoing research with a focus on program specification may be witnessed from the wealth of material collected in [3,23]. The map calculus [4] shows that these methods determine an active line of research in Logic.

This paper deals with stochastic rather than set-valued relations, it studies the converse of such a relation. It investigates furthermore some similarities between forming the converse for set-theoretic relations and for their stochastic cousins.

For introducing into the problem, let R be a relation, i.e., a set of pairs of, say, states. If $\langle x, y\rangle \in R$, then this is written as $x \to_R y$ and interpreted as a state transition from x to y. The converse $R^\smile$ shifts attention to the goal of the transition: $y \to_{R^\smile} x$ is interpreted as y being the goal of a transition from x. Now let $p(x, y)$ be the probability that there is a transition from x to y, and the question arises with which probability state y is the goal of a transition from x. This question cannot be answered unless we know the initial

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probabilities for the states. Then we can calculate $p_\mu^\smile(y, x)$ as the probability to make a transition from x to y weighted by the probability to start from x conditional to the event to reach y at all, i.e.

$$p_\mu^\smile(y, x) := \frac{\mu(x) \cdot p(x, y)}{\sum_t \mu(t) \cdot p(t, y)}.$$

Consider as an example the simple transition system p on three states given in the left hand side of Fig. 1. The converse $p_\mu^\smile$ for the initial probability $\mu := [1/2 \;\; 1/4 \;\; 1/4]$ is given on the right hand side.

The transition probabilities p are given through

$$\begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/5 & 1/2 & 3/10 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

with initial probabilities according to the stochastic vector $\mu := [1/2 \;\; 1/4 \;\; 1/4]$. The converse $p_\mu^\smile$ is then computed as

$$\begin{bmatrix} \frac{15}{31} & \frac{6}{31} & \frac{10}{31} \\ \frac{6}{11} & \frac{3}{11} & \frac{2}{11} \\ \frac{15}{34} & \frac{9}{34} & \frac{5}{17} \end{bmatrix}.$$

The situation is more complicated in the nonfinite case, which is considered here; since some measure theoretic constructions do not work in the general case, we assume that the measurable structure comes from Polish, i.e., second countable and completely metrizable topological spaces (like the real line $\mathbb{R}$). A definition of the converse $K_\mu^\smile$ of a stochastic relation K given an initial distribution μ is proposed in terms of disintegration. An interpretation of the converse in terms of random variables is given, and it is shown that the converse behaves with respect to composition like its set-theoretic counterpart, viz., $(K; L)_\mu^\smile = L_{K^\bullet(\mu)}^\smile; K_\mu^\smile$, where $K^\bullet(\mu)$ denotes the image distribution of μ under K, and the composition is the Kleisli composition for the corresponding monad (Section 4). This is of course the probabilistic counterpart to the corresponding law for relations R and S, which reads $(R; S)^\smile = S^\smile; R^\smile$.

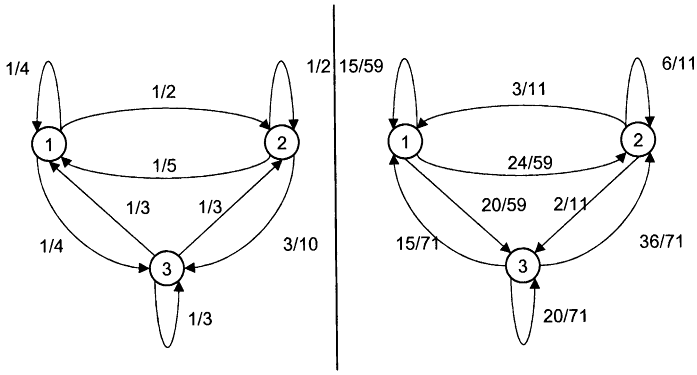

Figure 1. A stochastic relation and its converse.