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Syntax vs. semantics: A polarized approach

Olivier Laurent

Preuves Programmes Systèmes, CNRS – Université Paris VII, UMR 7126 – Case 7014, 2 place Jussieu, 75251 Paris Cedex 05, France

Abstract

We present a notion of sliced proof-nets for the polarized fragment of Linear Logic and a corresponding game model. We show that the connection between them is very strong through an equivalence of categories (this contains soundness, full completeness and faithful completeness).

Keywords: Linear logic; Game semantics; Proof-nets; Polarization; Full and faithful completeness

An important topic in the recent developments of denotational semantics has been the quest for stronger and stronger connections between the syntactical systems and the denotational models. Work towards bringing the two notions closer has come from both sides, and can be seen as an attempt to solve the general question "*what is a proof?*".

Full abstraction and full completeness (see [1,8]) results have been initiated with game semantics [1,2,15] and come with models containing only elements definable by the syntax. These results have been mainly obtained in the last 10 years for fragments of linear logic (for example MLL with and without MIX [1,14,25,5,6], MALL [3], ILL [19], LLP [21], ...) and for extensions of PCF (for example PCF [2,15], μPCF [18], Idealized Algol [4], ...). This full completeness property can be considered as a measurement of the precision of the semantics (whatever the syntax might be).

On the other side, the syntactical settings for logical systems have evolved progressively: sequent calculus, natural deduction, proof-nets. Although natural deduction is satisfactory for intuitionistic logic with $\to$, $\wedge$ and $\forall$, proof-nets permit intrinsic syntactical presentations of richer systems. Much work has been done since the original version [7] to remove sequentiality (boxes) and to make them more canonical. A way to evaluate the precision

E-mail address: Olivier.Laurent@pps.jussieu.fr.

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of a given syntax is to compare it with another one and to show that it realizes a quotient. Another approach is to use semantical means to see the identifications realized by a given model and that are not present in the syntax. In the spirit of a tight connection between syntax and semantics, the perfect case would correspond to an injective interpretation of the syntax in the model. The main results in this direction are due to Tortora de Falco [30] for fragments of linear logic with respect to coherent semantics. Work on full completeness led also to such faithfulness results (see [1,2,15]) but correspond to quite particular cases: MLL (and MLL with MIX) or λ-calculus. The extension of faithfulness results to the additive connectives was a very open question. Proof-nets for MALL recently given by Hughes and van Glabbeek [13] are very likely to lead to faithfulness results with respect to various models such as coherence spaces [7] or game semantics [3]. A solution for a polarized setting has also been given with proof-nets [24] but with a restriction on the use of the exponential connectives. These faithfulness results may be considered as a property of the syntax comparable to Böhm's theorem and more generally separation theorems. These syntactical theorems are about separation of terms by contexts. Here the separation is based on semantics but these two points of view sometimes coincide in particular in realizability models where terms are precisely interpreted by a set of accepting contexts (see for example Krivine's classical realizability [17] or Girard's ludics [11]).

In the spirit of Girard's program [10] to remove the distinction between syntax and semantics, this paper describes a strict correspondence between the polarized propositional fragment of linear logic $\mathsf{LL}_{\mathrm{pol}}$ [20] (with all the connectives and the units) and a polarized game model. Combining work from the syntactical side (sliced proof-nets) and work from the semantical side (game semantics), we prove that we arrive to a meeting point for the polarized framework. The notion of proof-nets we use is mainly the one described in [24] and our game model is a simplified version of [21] for $\mathsf{LL}_{\mathrm{pol}}$, also used in [22], which is enriched with variables. The polarization constraint we deal with here is not too strong since our setting is expressive enough to encode propositional classical logic with all connectives [23,20] (using a variation on Girard's embedding of intuitionistic logic into linear logic).

The results we prove in this paper can be summarized in categorical terms through an equivalence of categories between the syntax and the game model:

- Each object of the model is isomorphic to the interpretation of a formula.
- Each morphism of the model is the interpretation of a proof.
- The interpretations of two proofs are the same *iff* these proofs are $\beta\eta$-equal (with a canonical representative in each class of $\beta\eta$-equivalence given by cut-free sliced proof-nets).

We do not claim that this paper contains completely new ideas and structures. It is mainly a nice combination of (almost) known objects in order to get a precise comparison between them. The two main really new ingredients are the extension of the game model to variables with full completeness and the proof of the faithfulness result.

1. Polarized linear logic and proof-nets

The syntactical objects we are interested in are polarized sequent calculus proofs and polarized proof-nets. We first present the sequent calculus $\mathsf{LL}_{\mathrm{pol}}$ based on a restriction of LL