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Counting models for 2SAT and 3SAT formulae

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Abstract

We here present algorithms for counting models and max-weight models for 2SAT and 3SAT formulae. They use polynomial space and run in $O(1.2561^n)$ and $O(1.6737^n)$ time, respectively, where n is the number of variables. This is faster than the previously best algorithms for counting non-weighted models for 2SAT and 3SAT, which run in $O(1.3247^n)$ and $O(1.6894^n)$ time, respectively. In order to prove these time bounds, we develop new measures of formula complexity, allowing us to conveniently analyze the effects of certain factors with a large impact on the total running time. We also provide an algorithm for the restricted case of separable 2SAT formulae, with fast running times for well-studied input classes. For all three algorithms we present interesting applications, such as computing the permanent of sparse 0/1 matrices.

Keywords: Counting models; Satisfiability; Exponential-time algorithms; Exact algorithms; Upper bounds

1. Introduction

Most of the efforts in algorithm construction have been dedicated to algorithms for decision problems, i.e., finding *a* solution to the problem instance. For instance, this can

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involve finding a shortest path in a graph or a satisfying assignment to a boolean formula. As a natural extension we have the counting problems, where one wants to not merely decide the existence of a solution, but to find the *number* of solutions. One of the first algorithms for a counting problem came in the early 1960s with Ryser's [17] $O(n^2 2^n)$ time algorithm for counting the number of perfect matchings in a bipartite graph (also known as computing the *permanent* of a 0/1 matrix). In the 1970s Valiant [19] proposed the counting complexity class #P and showed that computing the permanent is complete for #P. It is interesting to note that both NP-complete problems as well as some decision problems, known to be in P, can have a counting counterpart which is #P-complete. For instance, both #2SAT and #3SAT are #P-complete [12,20].

Algorithms for #2SAT and #3SAT with better time bounds than the trivial $O(2^n)$ bound have been presented by Dubois [11], Zhang [21], Littman et al. [15] and Dahllöf et al. [8]. In this paper, which extends the work and results in [7,8], we improve on the previously best running times for both these problems, with algorithms that solve the more general weighted versions. Considering weights of solutions opens the field for more applications, as seen for instance in [3] and later in this paper.

For the problem of counting the number of max weight models to a 2SAT formula, here referred to as #2SAT$_w$, we present an algorithm with a running time in $O(1.2561^n)$, significantly improving on the previously best bound for #2SAT of $O(1.3247^n)$, achieved in [8]. There are several factors behind this improvement. One is a trick that allows us to split a constraint graph into its biconnected components. Among other things, this provides a way to remove variables which occur only once in a formula in polynomial time. Another factor is our method of analysis, where we use a special measure of formula complexity combining the number of variables and the number of clauses into a single value which is more representative of formula complexity than the standard n =#variables. By using this measure, we are able to divide our analysis into cases depending on the average degree of a formula, and capture and quantify the beneficial effects of having an algorithm which ensures that the average degree of a formula will be gradually decreasing. To say something about the corresponding decision problem, we see that it is not 2SAT , but rather a weighted variant, 2SAT$_w$. We are not aware of any dedicated algorithms for this problem, but to get some idea of its hardness, one can note that it contains MAXIMUM INDEPENDENT SET as a special case. MAXIMUM INDEPENDENT SET is known to be NP-complete and the so-far fastest poly-space algorithm runs in $O(1.2025^n)$ time, achieved by Robson [16].

For #3SAT$_w$, our algorithm has a running time in $O(1.6737^n)$, and the previously best result for #3SAT is $O(1.6894^n)$, achieved in [8]. This improvement is mainly due to a more precise complexity analysis, where we use another measure of formula complexity to better capture the effects of having clauses of cardinality 2 in the formula. As for the corresponding decision problem, we are not aware of any non-trivial worst-case time bounds for 3SAT$_w$ or any related optimization problems, but one can note that the so-far best exact poly-space algorithm for 3SAT runs in $O(1.4802^n)$ time.

We also present an algorithm that counts weighted models for 2SAT formulae having separable constraint graphs. While this class of formulae may sound exotic, we will present interesting graph applications. The separable graphs form a broad class, including many well-studied sub-classes such as the geometric graphs, graphs embeddable on surfaces of bounded genus, planar graphs, forests, grids, graphs with an excluded minor and graphs