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On the existence of equilibria in games with arbitrary strategy spaces and preferences ☆

Guoqiang Tian *

Department of Economics, Texas A&M University, College Station, TX 77843, USA
School of Economics, Shanghai University of Finance and Economics, Shanghai, 200433, China

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ABSTRACT

This paper provides necessary and sufficient conditions for the existence of pure strategy Nash equilibria by replacing the assumptions concerning continuity and quasiconcavity with a unique condition, passing strategy space from topological vector spaces to arbitrary topological spaces. Preferences may also be nontotal/nontransitive, discontinuous, nonconvex, or nonmonotonic. We define a single condition, *recursive diagonal transfer continuity (RDTC)* for aggregator payoff function and *recursive weak transfer quasi-continuity (RWTQC)* for individuals' preferences, respectively, which establishes the existence of pure strategy Nash equilibria in games with arbitrary (topological) strategy spaces and preferences without imposing any kind of quasiconcavity-related conditions.

1. Introduction

The notion of Nash equilibrium is probably one of the most important solution concepts in economics in general and game theory in particular, which has wide applications in almost all areas of economics as well as in business and other social sciences. The classical existence theorems on Nash equilibrium (e.g. in Nash, 1950, 1951; Debreu, 1952; Glicksberg, 1952; Nikaido and Isoda, 1955) typically assume *continuity* and *quasiconcavity* for the payoff functions, in addition to convexity and compactness of strategy spaces, which require strategy spaces be topological vector spaces. However, in many important economic models, such as those in Bertrand, 1883, Hotelling (1929), Dasgupta and Maskin (1986), and Jackson (2009), payoffs are discontinuous and/or non-quasiconcave, and strategy spaces are nonconvex and/or noncompact.

Accordingly, economists continually strive to seek weaker conditions that can guarantee the existence of equilibrium. Some seek to weaken the quasiconcavity of payoffs or substitute it with some types of transitivity/monotonicity of payoffs (cf. McManus, 1964; Roberts and Sonnenschein, 1977; Topkis, 1979; Nishimura and Friedman, 1981; Milgrom and Roberts, 1990; Vives, 1990), some seek to weaken the continuity of payoff functions (cf. Dasgupta and Maskin, 1986; Simon, 1987; Simon and Zame, 1990; Tian, 1992a,b,c, 1994; Reny, 1999, 2009; Bagh and Jofre, 2006; Monteiro and Page, 2007; Morgan and Scalzo, 2007; Nessah and Tian, 2008; Carmona, 2009, 2011; Scalzo, 2010; Balder, 2011; Prokopovych, 2011, 2013), while others seek to weaken both quasiconcavity and continuity (cf. Baye et al., 1993, de Castro, 2011, McLennan et al., 2011; Barelli and Meneghel, 2013).

However, all the existing results are under the assumption of topological vector spaces, impose linear (convex) or lattice structures and only provide sufficient conditions for the existence of equilibrium.[1] In order to apply a fixed-point theorem (say, Brouwer, Browder, Kakutani, Michael, or Knaster, Kuratowski, and Mazurkiewicz, etc.), they all need to assume some forms of quasiconcavity[2] (or transitivity/monotonicity) and continuity of payoffs,

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* Correspondence to: Department of Economics, Texas A&M University, College Station, TX 77843, USA.

E-mail address: gtian@tamu.edu.

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[1] McLennan et al. (2011) and Barelli and Meneghel (2013) recently provide necessary and sufficient conditions for the existence of Nash equilibrium. However, they obtain their existence results under the linear structures.

[2] For mixed strategy Nash equilibrium, quasiconcavity is automatically satisfied since the mixed extension has linear payoff functions. Thus only some form of continuity matters for the existence of mixed strategy Nash equilibrium.

in addition to compactness and convexity of strategy space. While it may be the convex structure that easily connects economics to mathematics, in many important situations where commodities or alternatives are invisible so that the choice spaces are discrete, there are no convex/lattice structures.

Thus, convexity assumption excludes the possibility of considering discrete games, and consequently seriously limits the applicability of economic theory. As such, the intrinsic nature of equilibrium has not been fully understood. Why does or does not a game have an equilibrium? Are continuity and quasiconcavity both essential to the existence of equilibrium? If so, can continuity and quasiconcavity be combined into one single condition? One can easily find simple examples of economic games that have or do not have an equilibrium (see Examples 3.1 and 3.2), but none of them can be used to reveal the existence/non-existence of equilibria in these games. This paper sheds some light on these questions.

We fully characterize the existence of pure strategy Nash equilibrium in general games with arbitrary topological strategy spaces[3] that may be discrete or non-convex and payoffs (resp. preferences) that may be discontinuous (resp. discontinuous or nontotal/nontransitive) or do not have any form of quasi-concavity (resp. convexity) or monotonicity. We introduce the notions of recursive transfer continuities, specifically *recursive diagonal transfer continuity (RDTC)* for aggregator payoff function and *recursive weak transfer quasi-continuity (RWTQC)* for individuals' preferences.

It is shown that the single condition, RDTC (resp. RWTQC) is necessary, and further, under compactness of strategy space, sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary strategy spaces and payoffs (resp. preferences).[4] We also provide an existence theorem for a strategy space that may not be compact. We show that RDTC (resp. RWTQC) with respect to a compact set is necessary and sufficient for the existence of pure strategy Nash equilibrium in games with arbitrary (topological) strategy spaces and general payoffs (resp. preferences). RDTC (resp. RWTQC) also permits the existence of symmetric pure strategy Nash equilibria in games with general strategy spaces and payoffs (resp. preferences).

RDTC (resp. RWTQC) strengthens diagonal transfer continuity introduced in Baye et al. (1993) (resp. weak transfer quasi-continuity introduced in Nessah and Tian, 2008) to allow recursive (sequential) transfers in order to get rid of the diagonal transfer quasiconcavity assumption (resp. the strong diagonal transfer quasiconcavity assumption) so that these conditions turn out to be necessary and sufficient for the existence of equilibria in compact games. As such, no quasiconcavity/monotonicity-related conditions are assumed. These results may be used to argue the existence of equilibrium in general games with no linear (convex) structures such as equilibrium issues in market design theory and matching theory. In the paper, we also provide sufficient conditions for the existence of equilibrium without imposing any form of quasiconcavity.

The remainder of the paper is organized as follows. Section 2 provides basic notation and definitions, and analyzes the essence of Nash equilibrium. Section 3 investigates the existence of pure strategy Nash equilibrium by using aggregate payoffs and individuals' preferences respectively. We also provide sufficient conditions for recursive transfer continuities. Section 4 extends the results to symmetric pure strategy Nash equilibrium. Concluding remarks are offered in Section 5.

[3] In particular, the strategy spaces may not be metrizable, locally convex, Hausdorff, or even not regular.

[4] As such, one *cannot* say that RDTC (resp. RWTQC) is equivalent to Nash equilibrium.

2. Preliminaries: Nash equilibrium and its intrinsic nature

2.1. Notions and definitions

Let I be the set of players that is either finite or countably infinite. Each player i's strategy space X_i is a general topological space that may not be metrizable, locally convex, Hausdorff, or even not regular. Denote by $X = \prod_{i\in I} X_i$ the Cartesian product of the sets of strategy profiles, equipped with the product topology. For each player $i \in I$, denote by $-i$ all other players rather than player i. Also denote by $X_{-i} = \prod_{j\neq i} X_j$ the Cartesian product of the sets of strategies of players $-i$. Without loss of generality, assume that player i's preference relation is given by the weak preference $\succcurlyeq_i$ defined on X, which may be nontotal or nontransitive.[5] Let $\succ_i$ denote the asymmetric part of $\succcurlyeq_i$, i.e., $y \succ_i x$ if and only if $y \succcurlyeq_i x$ but not $x \succcurlyeq_i y$.

A *game* $G = (X_i, \succcurlyeq_i)_{i\in I}$ is simply a family of ordered tuples $(X_i, \succcurlyeq_i)$.

When $\succcurlyeq_i$ can be represented by a payoff function $u_i : X \to \mathbb{R}$, the game $G = (X_i, u_i)_{i\in I}$ is a special case of $G = (X_i, \succcurlyeq_i)_{i\in I}$.

A strategy profile $x^* \in X$ is a *pure strategy Nash equilibrium* of a game G iff,

$$x^* \succcurlyeq_i (y_i, x^*_{-i}) \quad \forall i \in I,\ \forall y_i \in X_i.$$

2.2. The essence of equilibrium and why the existing results are only sufficient

Before proceeding to the notions of recursive transfer continuities, we first analyze the intrinsic nature of Nash equilibrium, and why the conventional continuity is unnecessarily strong and most of the existing results provide only sufficient but not necessary conditions.

In doing so, we define an "upsetting" (irreflexive) binary relation on X, denoted by $\succ$ as follows:

$$y \succ x \quad \text{iff} \quad \exists i \in I \quad s.t.\ (y_i, x_{-i}) \succ_i x. \tag{1}$$

In this case, we say strategy profile y *upsets* strategy profile x. It is clear that "$y \succ x$ for $x, y \in X$" is equivalent to "$x \in X$ is not an equilibrium". We will use these terms interchangeably. Then, one can easily see that a strategy profile $x^* \in X$ is a pure strategy Nash equilibrium if and only if there does not exist any strategy y in X that upsets x^*.

When $x \in X$ is not a pure strategy Nash equilibrium, then there exists a strategy profile $y \in X$ such that $y \succ x$. To establish the existence of an equilibrium, it usually requires *all* strategies in a neighborhood $\mathcal{V}_x$ of x be upset by some strategy profile $z \in X$, denoted by $z \succ \mathcal{V}_x$, i.e., $z \succ x'$ for all $x' \in \mathcal{V}_x$. The topological structure of the conventional continuity surely secures this upsetting relation locally at x by y, i.e., there always exists a neighborhood $\mathcal{V}_x$ of x such that $y \succ \mathcal{V}_x$. As such, no transfers (say, from y to z) or switchings (from player i to j) are needed for securing this upsetting relation locally at x. However, when u_i is not continuous, such a topological relation between the upsetting point y and the neighborhood $\mathcal{V}_x$ may no longer be true, i.e., we may not have $y \succ \mathcal{V}_x$. But, if y can be transferred to z so that $z \succ \mathcal{V}_x$, then the upsetting relation $\succ$ can be secured locally at x. This naturally leads to the following notion of transfer continuity, which is a weak notion of continuity

[5] The results obtained for weak preferences $\succcurlyeq_i$ can also be used to get the results for strict preferences $\succ_i$. Indeed, from $\succ_i$, we can define a weak preference $\succcurlyeq_i$ on $X \times X$ as follows: $y \succcurlyeq_i x$ if and only if $\neg x \succ_i y$. The preference $\succcurlyeq_i$ defined in such a way is called the completion of $\succ_i$. A preference $\succcurlyeq_i$ is said to be complete iff, for any $x, y \in X$, either $x \succcurlyeq_i y$ or $y \succcurlyeq_i x$. A preference $\succcurlyeq_i$ is said to be total iff, for any $x, y \in X$, $x \neq y$ implies $x \succcurlyeq_i y_i$ or $y \succcurlyeq_i x$.