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Differentiability of von Neumann–Morgenstern utility functions

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ABSTRACT

This paper studies necessary and sufficient preference-based conditions for differentiability of risk averse (prudent, or temperate) von Neumann–Morgenstern utility functions. The very idea to devise those conditions is based on the reverse claim of an old observation by Arrow that a risk-averse expected-utility maximizer will always accept a sufficiently small stake in any positive expected-value bet if her von Neumann–Morgenstern utility function is differentiable.

1. Introduction

The expected utility theory originated by von Neumann and Morgenstern (1944) is an indispensable tool in economic analyses under risk. Preference structures to guarantee the existence of a von Neumann–Morgenstern (vNM) utility function are well known under various structural assumptions, since they have been extensively studied and generalized in the last half century (see, for example, Fishburn, 1970, 1982, Hammond, 1999). In addition, continuity, concavity, and differentiability of the function are also important analytical properties in economic applications.

Conditions which imply continuity and concavity can be stated in terms of decision maker's preferences for risks, i.e., simple probability distributions over the final wealth levels. On the other hand, although differentiability of vNM utility functions is commonly imposed as a technically convenient assumption in economic applications, there seems to be no attempt to examine preference-based characterizations except Nielsen (1999). However, his conditions are indirectly formulated by preferences through risk and probability premia for small risks.

This paper studies higher-order differentiabilities up to three times. We develop necessary and sufficient preference-based conditions for differentiability of risk averse (prudent or temperate) vNM utility functions. The very idea to devise our differentiability conditions is based on the reverse claim of the following old observation by Arrow (1971): a risk-averse expected-utility maximizer will always accept a sufficiently small stake in any positive expected-value bet when her vNM utility function is differentiable. Although it is well known that a concave (or convex) function defined on a real interval is differentiable on all points perhaps except at most countable points in the domain, I believe that it would be of interest and a theoretical contribution in decision making under risk to uncover behavioral conditions for differentiability of vNM utility functions which have long been ignored or no one simply came across in the literature.

The paper is organized as follows. In Section 2, the first characterization, which is Arrow's reverse claim, saying that the decision maker exhibits preference for small positive risk taking, is introduced and discussed. When the decision maker's vNM utility function u is strictly increasing and concave (i.e., risk averse), it is proved that u is differentiable. Section 3 assumes that the first derivative u' of u exists and that the decision maker is prudent in the sense that u' is strictly decreasing and convex. Then we present and discuss the second characterization, in which the decision maker is said to exhibit preference for downward shift of small positive risks. For the prudent decision maker with the first derivative u', we prove that the second derivative u'' exists and is increasing. In Section 4, it is assumed that the second derivative u'' of u exists and that the decision maker is temperate in the following sense: u'' is strictly increasing and concave. Then the third characterization is devised and discussed so that the decision maker exhibits preference for downward augmentation of small positive risks. For the temperate decision maker with the second derivative u'', it is proved that the third derivative u''' exists and is decreasing. Section 5 concludes the paper.

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2. Differentiability

Throughout the paper, let u be a strictly increasing and continuous vNM utility function over an open real interval $I \subseteq \mathbb{R}$. Each $x \in I$ is interpreted as a decision maker's wealth level. Random variables are measurable functions on an algebra of subsets of a state space S. By $\mathcal{R}$, we shall denote the set of all "simple" random variables, or *risks*, which take on a finite number of wealth levels in I. For risk $\widetilde{x} \in \mathcal{R}$, let $P_{\widetilde{x}}$ denote its induced probability distribution function on I, so the support $\{x \in I : P_{\widetilde{x}}(x) > 0\}$ of $P_{\widetilde{x}}$ is finite. Given a sufficiently rich set S, it is assumed throughout that any probability distribution function with a finite support can be generated by a risk in $\mathcal{R}$.

A random variable is *nondegenerate* if its support is not a singleton. Each $x \in I$ will be identified with the degenerate risk that yields wealth level x with probability one. By $\langle \widetilde{x}, \widetilde{y} \rangle$, we shall denote a 50–50 risk that yields risk $\widetilde{x}$ with probability one-half and risk $\widetilde{y}$ otherwise. Addition and positive-scalar multiplication of risks are defined as follows. When we add two risks, we shall always assume that they are statistically independent, i.e., for $\widetilde{x}, \widetilde{y} \in \mathcal{R}$, $\widetilde{x} + \widetilde{y}$ means a risk that yields wealth level $x + y$ with probability $P_{\widetilde{x}}(x)P_{\widetilde{y}}(y)$. In particular, $\widetilde{x} + w$ for $w \in I$ is a risk which augments the final wealth level from the sure amount w by the risk $\widetilde{x}$. It may be interpreted as taking risk $\widetilde{x}$ at wealth level w. For $\epsilon > 0$, a positive-scalar multiplication $\epsilon\widetilde{x}$ of risk $\widetilde{x}$, called a *proportional fraction* of $\widetilde{x}$, is defined to be a risk that yields wealth level ϵx with probability $P_{\widetilde{x}}(x)$. When $0 < \epsilon < 1$, $\epsilon\widetilde{x}$ is said to be a *small* proportional fraction of $\widetilde{x}$. Do not confuse $\widetilde{x}+\widetilde{x}$ with $2\widetilde{x}$, where those are different risks by definition. We shall assume that all risks and their additions appeared in what follows take values in I.

Expected value and expected utility of $\widetilde{x} \in \mathcal{R}$ w.r.t. u are respectively given by

$$E\widetilde{x} = \sum_x x P_{\widetilde{x}}(x),$$

$$Eu(\widetilde{x}) = \sum_x u(x) P_{\widetilde{x}}(x).$$

We say that a risk $\widetilde{x}$ is *fair* (resp., *positive*, or *negative*) if $E\widetilde{x} = 0$ (resp., $E\widetilde{x} > 0$, or $E\widetilde{x} < 0$). Define

$$\mathcal{R}^0 = \{\widetilde{x} \in \mathcal{R} : E\widetilde{x} = 0\},$$

$$\mathcal{R}^+ = \{\widetilde{x} \in \mathcal{R} : E\widetilde{x} > 0\}.$$

Let $\succ$ be an asymmetric binary relation on $\mathcal{R}$, read as "is preferred to." Thus, $\widetilde{x} \succ \widetilde{y}$ means that wealth level determined by risk $\widetilde{x}$ is preferred to wealth level determined by $\widetilde{y}$. Also, for example, $\widetilde{x} + w \succ \widetilde{y} + w$ may read as "taking risk $\widetilde{x}$ at wealth level w is preferred to taking risk $\widetilde{y}$ at wealth level w." Weak preference and indifference relations, respectively denoted by $\succsim$ and $\sim$, are defined as usual: for all $\widetilde{x}, \widetilde{y} \in \mathcal{R}$, $\widetilde{x} \succsim \widetilde{y}$ if $\neg(\widetilde{y} \succ \widetilde{x})$; $\widetilde{x} \sim \widetilde{y}$ if $\neg(\widetilde{x} \succ \widetilde{y})$ and $\neg(\widetilde{y} \succ \widetilde{x})$.

Throughout the paper, we shall assume that $\succ$ on $\mathcal{R}$ is represented by maximization of expected utilities w.r.t. u, i.e., for all $\widetilde{x}, \widetilde{y} \in \mathcal{R}$ and all $w \in I$,

$$\widetilde{x} + w \succ \widetilde{y} + w \iff Eu(\widetilde{x} + w) > Eu(\widetilde{y} + w).$$

That is, taking risk $\widetilde{x}$ at wealth level w is preferred to taking risk $\widetilde{y}$ at wealth level w if and only if expected utility of $\widetilde{x} + w$ is greater than expected utility of $\widetilde{y} + w$.

In what follows, under the assumptions introduced above, we shall impose two additional conditions on $(\mathcal{R}, \succ)$ to guarantee differentiability of u. In those conditions, we are concerned with which risks in $\mathcal{R}$ the decision maker prefers to (or not to) take at a wealth level. The first condition we shall impose on $(\mathcal{R}, \succ)$ is the well-known property of risk aversion, (i.e., aversion to fair risk taking), which asserts that the decision maker prefers not to take any nondegenerate fair risk at any wealth level. This is stated as follows.

Axiom A1 (*Aversion to Fair Risk Taking*). For all nondegenerate $\widetilde{x} \in \mathcal{R}^0$ and all $w \in I$, $w \succ \widetilde{x} + w$.

The well-known implication of this axiom is that the vNM utility function u is concave (in the strict sense throughout). Of course, conversely, $(\mathcal{R}, \succ)$ with u concave exhibits aversion to fair risk taking.

We note that, assuming aversion to fair risk taking, every negative risk must be preferred not to take at any wealth level. To see this, consider any nondegenerate $\widetilde{x} \in \mathcal{R}$ with $E\widetilde{x} < 0$. Then $E\widetilde{x} + \delta = 0$ for some $\delta > 0$. By aversion to fair risk taking, $w \succ \widetilde{x} + \delta + w$. Since u is strictly increasing, it follows from the first order stochastic dominance that $\widetilde{x} + \delta + w \succ \widetilde{x} + w$. Thus $w \succ \widetilde{x} + w$, so $\widetilde{x}$ is preferred not to take at any wealth level w. The degenerate case is obvious.

On the other hand, any *trivial* positive risk $\widetilde{x}$ with $\widetilde{x} \geq 0$, meaning that $x \geq 0$ whenever $P_{\widetilde{x}}(x) > 0$, is preferred to take at any wealth level. For a nontrivial positive risk $\widetilde{x}$ with $P_{\widetilde{x}}(x) > 0$ for some $x < 0$, however, the situation is blurred because its preferability at a wealth level depends on u. The following theorem clarifies the situation.

Theorem 2.1. (1) *If a positive risk $\widetilde{x}$ is preferred to take at a wealth level $w \in I$, then so are all small proportional fractions of the risk at w, whenever u is concave.*

(2) *Any positive risk is preferred to take at any wealth level and any negative risk is preferred not to take at any wealth level if and only if u is a linear function.*

Proof. (1) Suppose that u is concave. Take an $\widetilde{x} \in \mathcal{R}^+$. Assume that $\widetilde{x}+w \succ w$. Then $Eu(\widetilde{x}+w)-u(w) > 0$. Since u is concave, it follows that, for all $x \in I$ and all $0 < \epsilon < 1$,

$$u(\epsilon x + w) - u(w) > \epsilon\,(u(x + w) - u(w))\,.$$

Thus

$$Eu(\epsilon\widetilde{x} + w) - u(w) > \epsilon\,[Eu(\widetilde{x} + w) - u(w)] > 0,$$

so that $Eu(\epsilon\widetilde{x} + w) > u(w)$. Hence any small proportional fraction $\epsilon\widetilde{x}$ of $\widetilde{x}$ is preferred to take at wealth level w.

(2) Sufficiency of linearity of u is easily obtained. Thus we show its necessity. Suppose that any positive risk is preferred to take at any wealth level and that any negative risk is preferred not to take at any wealth level. Take any $\widetilde{x} \in \mathcal{R}^0$. Then $E(\widetilde{x} + \delta) > 0$ for any $\delta > 0$. For any $w \in I$, it follows from the assumption that

$$\widetilde{x} + \delta + w \succ w,$$

so that $Eu(\widetilde{x} + \delta + w) > u(w)$. If $Eu(\widetilde{x} + w) < u(w)$, then it follows from continuity and monotonicity of u that, for some $\epsilon > 0$,

$$Eu(\widetilde{x} + \epsilon + w) < u(w).$$

This contradicts the assumption. Hence $Eu(\widetilde{x} + w) \geq u(w)$. Similarly, for any $\delta > 0$, $w \succ \widetilde{x} - \delta + w$, so that $u(w) > Eu(\widetilde{x} - \delta + w)$. If $Eu(\widetilde{x} + w) > u(w)$, then $Eu(\widetilde{x} - \epsilon + w) > u(w)$ for some $\epsilon > 0$, a contradiction. Therefore, $Eu(\widetilde{x} + w) \leq u(w)$. Hence we conclude that $Eu(\widetilde{x} + w) = u(w)$. Since $\widetilde{x} \in \mathcal{R}^0$ and w are arbitrary, u must be linear. □

It follows from Theorem 2.1 that, if u is not linear but concave, then there exists a positive risk that is preferred not to take at some wealth level. At this point, we do not know whether or not some small proportional fraction of such a positive risk is preferred to take at the wealth level. The following axiom requires, however, that, for any positive risk and any wealth level w, some small proportional fraction of the risk must be always preferred to take at w.

Axiom A2 (*Preference for Small Positive Risk Taking*). For all $\widetilde{x} \in \mathcal{R}^+$ and all $w \in I$, there exists a $0 < \epsilon < 1$ such that $\epsilon\widetilde{x} + w \succ w$.