Journal of Mathematical Economics 60 (2015) 110–114

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

On axiomatizations of the Shapley value for assignment games[☆]

René van den Brink[a,*], Miklós Pintér[b,c,d]

[a] VU University, Department of Econometrics and Tinbergen Institute, De Boelelaan 1105, 1081 HV Amsterdam, The Netherlands
[b] Corvinus University of Budapest, Department of Mathematics, 1093, Budapest, Fovám tér 13-15, Hungary
[c] MTA-BCE "Lendület" Strategic Interactions Research Group, 1093, Budapest, Fovám tér 13-15, Hungary
[d] Department of Business and Economics, University of Pecs, Hungary

ARTICLE INFO

Article history:
Received 14 March 2013
Received in revised form
21 November 2013
Accepted 26 June 2015
Available online 7 July 2015

Keywords:
Game theory
Assignment game
Shapley value
Graph game
Submarket efficiency
Valuation fairness

ABSTRACT

We consider the problem of axiomatizing the Shapley value on the class of assignment games. It turns out that several axiomatizations of the Shapley value on the class of all TU-games do not characterize this solution on the class of assignment games. However, when considering an assignment game as a (communication) graph game where the game is simply the assignment game and the graph is a corresponding bipartite graph where buyers (sellers) are connected with sellers (buyers) only, we show that Myerson's component efficiency and fairness axioms do characterize the Shapley value on the class of assignment games. Moreover, these two axioms have a natural interpretation for assignment games. Component efficiency yields submarket efficiency stating that the sum of the payoffs of all players in a submarket equals the worth of that submarket, where a submarket is a set of buyers and sellers such that all buyers in this set have zero valuation for the goods offered by the sellers outside the set, and all buyers outside the set have zero valuations for the goods offered by sellers inside the set. Fairness of the graph game solution boils down to valuation fairness stating that only changing the valuation of one particular buyer for the good offered by a particular seller changes the payoffs of this buyer and seller by the same amount.

1. Introduction

The history of *assignment games* goes back to the XIX century to Böhm-Bawerk's (1923) horse market model. Later Shapley and Shubik (1972) introduced the formal, modern concept of assignment games.

One of the most popular solution concepts for TU-games is the *Shapley value* (Shapley, 1953). Numerous axiomatizations of the Shapley value are known in the literature, for example (i) Shapley's original axiomatization (Shapley, 1953) by efficiency, the null player property (originally stated together as the carrier axiom), symmetry and additivity (also discussed by Dubey, 1975 and Peleg and Sudhölter, 2003), (ii) Young's (1985) axiomatization replacing additivity and the null player property by strong monotonicity (also discussed by Moulin (1988) and Pintér (2012)), (iii) Chun's (1991) replacing strong monotonicity by coalitional strategic equivalence, (iv) van den Brink's (2001) replacing (in Shapley's original axiomatization) additivity and symmetry by fairness, and (v) Hart and Mas-Colell's (1989) approaches using the potential function and a related reduced game consistency. It turns out that none of these characterizations are valid on the class of assignment games in the sense that they do not characterize a unique solution.

In this paper, we show that when considering an assignment game as a (communication) graph game where the game is simply the assignment game and the graph is a corresponding bipartite graph where buyers (sellers) are connected with sellers (buyers) only, Myerson's (1977) component efficiency and fairness axioms do characterize the Shapley value on the class of assignment games. Moreover, the axioms have a natural interpretation for these games.

An assignment game is fully described by the *assignment situation* being a set of buyers, a set of sellers, and for every buyer a valuation of the good offered by each seller. Instead of defining an assignment game as a graph game, we will directly work on the class of these assignment situations. For such assignment situations, component efficiency of a graph game solution boils down to *submarket efficiency* stating that the sum of the payoffs of all players in a submarket equals the worth of that submarket, where a *submarket* in an assignment situation is a set of buyers and sellers such that all buyers in this set have zero valuation for the goods

☆ We thank an anonymous referee for her/his useful comments. Miklós Pintér gratefully acknowledges the Financial support by the Hungarian Scientific Research Fund (OTKA) grant number K 101224, SPOR grant number 4.2.2.B-15/1/konv and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.
* Corresponding author.
E-mail addresses: jrbrink@feweb.vu.nl (R. van den Brink), miklos.pinter@uni-corvinus.hu (M. Pintér).

http://dx.doi.org/10.1016/j.jmateco.2015.06.016
0304-4068/

offered by the sellers outside the set, and all buyers outside the set have zero valuations for the goods offered by sellers inside the set.

Fairness of the graph game solution boils down to *valuation fairness* stating that only changing the valuation of one particular buyer for the good offered by a particular seller changes the payoffs of this buyer and seller by the same amount. We show that these two axioms do characterize the Shapley solution for assignment situations being the solution that is obtained by applying the Shapley value to the corresponding assignment game. So, we obtain a positive result by viewing an assignment game as a graph game.

Besides introducing and axiomatizing his solution, Myerson (1977) also shows that it is *stable* for superadditive graph games in the sense that two players never get worse off when building a link between them. The Shapley solution for assignment situations is *valuation monotonic* in the sense that the payoffs of a buyer i and a seller j do not decrease if only the valuation of buyer i for the good offered by seller j increases.

2. Preliminaries

2.1. TU-games

Let N be a non-empty, finite set, let $|N|$ be its cardinality, and let $\mathcal{P}(N)$ denote the power set of N. A *transferable utility* (TU) game with player set N is a pair (N, v) with *characteristic function* $v : \mathcal{P}(N) \to \mathbb{R}$ such that $v(\emptyset) = 0$. Since we take the player set N to be fixed, we represent a TU-game (N, v) simply by its characteristic function v. The class of all characteristic functions on player set N is denoted by $\mathcal{G}^N$.

Game v is superadditive if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ with $S \cap T = \emptyset$.

A (single-valued) *solution* on $\mathcal{C} \subseteq \mathcal{G}^N$, is a function $\phi : \mathcal{C} \to \mathbb{R}^N$.

In this paper we focus on the *Shapley value* (Shapley, 1953) being the solution $\phi^{Sh}: \mathcal{G}^N \to \mathbb{R}^N$, for every $v \in \mathcal{G}^N$, given by

$$\phi_i^{Sh}(v) = \sum_{T \subseteq N \setminus \{i\}} \frac{|T|!(|N \setminus T| - 1)!}{|N|!} m_i^T(v) \quad \text{for all } i \in N,$$

where, for any $v \in \mathcal{G}^N$, $i \in N$ and $T \subseteq N$, $m_i^T(v) = v(T \cup \{i\}) - v(T)$ is player i's marginal contribution to coalition T in game v. We refer to $\phi_i^{Sh}(v)$ as the Shapley value of player i in game $v \in \mathcal{G}^N$.

Several axiomatizations of the Shapley value can be found in the literature, such as the axiomatization[1] by Pareto optimality (also known as efficiency), the null player property, the equal treatment property and additivity (Shapley, 1953), Pareto optimality, the equal treatment property and strong monotonicity (Young, 1985), Pareto optimality, the equal treatment property and marginality (also by Young, 1985), Pareto optimality, the equal treatment property and coalitional strategic equivalence (Chun, 1991) and Pareto optimality, the null player property and fairness (van den Brink, 2001).[2]

2.2. Assignment games

Let $B, S \subseteq N$ be two non-empty sets such that $B \cap S = \emptyset$ and $B \cup S = N$. The interpretation is the following. The sets B and S are the sets of buyers and sellers, respectively. Every buyer wants one good, and every seller owns one good. These goods are not exactly the same, so a buyer can have different valuations for the goods owned by different sellers. We assume that the sellers have reservation value zero for every good. The nonnegative valuation (reservation value) of buyer $i \in B$ for the good offered by seller $j \in S$ is denoted by $a_{i,j} \geq 0$. So, buyer i and seller j can make a deal and earn worth $a_{i,j}$. Buyers cannot trade among each other (since they do not own a good), and also sellers cannot earn a worth among themselves since their valuation is zero.

Let A be the $|B| \times |S|$ non-negative matrix with $a_{i,j}$ its (i, j) element. We refer to this matrix A as an *assignment situation* or *valuation matrix* on (B, S). We denote the collection of all assignment situations on (B, S) by $\mathcal{A}^{B,S}$. Furthermore for all $T \subseteq N$, a *matching* on T is a set of sets $M \subseteq \{\{i, j\} \subseteq T \mid i \in B \cap T, j \in S \cap T\}$ such that for every $g \in T$, $|\{\{h, k\} \in M \mid g \in \{h, k\}\}| \leq 1$. So, buyers can only be matched with sellers, sellers can only be matched with buyers, and every buyer (seller) can be matched with at most one seller (buyer). Let $\mathcal{M}(T)$ be the set of all *matchings* of T. Taking the sets of buyers B and sellers S fixed, the *assignment game* (see Shapley and Shubik, 1972) for valuation matrix A is the game v_A on $N = B \cup S$, given by[3]

$$v_A(T) = \max_{M \in \mathcal{M}(T)} \sum_{\{i,j\} \in M} a_{i,j} \quad \text{for all } T \subseteq B \cup S.$$

The elements of

$$\arg\max_{M \in \mathcal{M}(T)} \sum_{\{i,j\} \in M} a_{i,j}$$

are called the *maximal matchings* of coalition T. For any set of buyers and sellers T, the worth of this coalition is the maximum aggregated worth of the deals the involved players can achieve contingent on every player trading with at most one other player from the other type.

Remark 2.1. Since in the definition of assignment games, B and S are non-empty, in this paper every assignment game has at least two players.

2.3. Graph games

Myerson (1977) introduced a model in which it is assumed that the players in a game v are part of a communication structure that is represented by an *undirected graph* (N, L), with the player set N as the set of nodes and $L \subseteq \{\{i, j\} \mid i, j \in N, i \neq j\}$ being a collection of *edges* or *links*, that is, subsets of N such that each element of L contains precisely two elements. Since in this paper the nodes in a graph represent the players in a game, we use the same notation for the set of nodes as the set of players, and refer to the nodes in a graph just as players.

If there is no confusion about the player set N, we denote a graph on N just by its set of links L and refer to this as the graph. We denote the class of all possible sets of links on N by $\mathcal{L}^N$. A sequence of k different nodes $(i_1, \ldots, i_k)$ is a *path* between players i_1 and i_k in $L \in \mathcal{L}^N$ if $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k - 1$. A coalition $S \subseteq N$ is *connected* in graph L if every pair of players in S is connected by a path that only contains players from S, that is, for every $i, j \in S$, $i \neq j$, there is a path $(i_1, \ldots, i_k)$ such that $i_1 = i$, $i_k = j$ and $\{i_1, \ldots, i_k\} \subseteq S$. Coalition $T \subseteq S$ is a *component* of S in graph L if it is a maximally connected subset of S, that is, T is connected in $L(S)$ and for every $h \in S \setminus T$ the coalition $T \cup \{h\}$ is not connected in $L(S)$, where $L(S) = \{\{i, j\} \in L \mid \{i, j\} \subseteq S\}$. We denote the set of components of $S \subseteq N$ in L by $C_L(S)$.

A pair $(v, L) \in \mathcal{G}^N \times \mathcal{L}^N$ is referred to as a *graph game* on N. Following Myerson (1977), in the graph game (v, L) players can cooperate if and only if they are able to communicate with each

[1] We refer the reader to the mentioned literature for the definition of the axioms.

[2] For games on variable player sets the Shapley value is characterized by, e.g. Pareto optimality, covariance, the equal treatment property and consistency in Hart and Mas-Colell (1989) who also expressed it by a potential.

[3] We use the convention that the empty sum is 0.